Stereographic Projection and its Importance*

Stereographic projection was invented for making maps of the earth and the celestial sphere. It is an angle preserving map of the sphere minus one point onto the Euclidean plane.
Complex differentiability (of maps from the plane to the plane) is the same as real differentiability plus preservation of oriented angles (except were the derivative vanishes). Angle preserving differentiable maps are therefore important for the theory of complex functions. The stereographic projection can be interpreted as mapping the 2-sphere to the Euclidean plane, compactified by a *point at infinity*. All the complex rational functions can be extended to *infinity*, i.e. they can be considered as differentiable, angle preserving maps from the 2-sphere to the 2-sphere. The stereographic projection turns this "can be considered as" into a simple explicit relation. The 2-sphere, in this context, is called the *Riemann Sphere* and the image plane is called the *Gaussian Plane*.

In 3D-XplorMath a 3D image of the stereographic projection can be reached via the last entry of the Action Menu in the Conformal Map Category. *Stereographic projection is defined as the central projection from one point on the sphere onto the opposite tangent plane.*

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

Formulas for the Stereographic Projection

Algebraically it is slightly more convenient to map the sphere not to the opposite tangent plane but to the parallel plane through the midpoint. More explicitly, the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is projected from $(0,0,-1)$ to the plane $\{z=0\}$ (and similar formulas work in all dimensions):

$$St(x,y,z) := \frac{1}{1+z} \cdot (x,y,0), \text{ where } x^2+y^2+z^2=1,$$

because the three points involved lie on a line:

$$\tfrac{1}{1+z} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \tfrac{z}{1+z} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{x}{(1+z)} \\ \frac{y}{(1+z)} \\ 0 \end{pmatrix} = St\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \\ 0 \end{pmatrix}.$$

The inverse map is given by:

$$St^{-1}(\xi,\eta) := \frac{1}{1+\xi^2+\eta^2} \cdot \begin{pmatrix} 2\xi \\ 2\eta \\ 1-\xi^2-\eta^2 \end{pmatrix} \in \mathbb{S}^2,$$

because, again, the three points involved lie on a line:

$$\tfrac{2}{1+\xi^2+\eta^2} \cdot \begin{pmatrix} \xi \\ \eta \\ 0 \end{pmatrix} + \tfrac{-1+\xi^2+\eta^2}{+1+\xi^2+\eta^2} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = St^{-1}(\xi,\eta) =$$

$$\tfrac{1}{1+\xi^2+\eta^2} \cdot \begin{pmatrix} 2\xi \\ 2\eta \\ 1-\xi^2-\eta^2 \end{pmatrix}.$$

Claim: Stereographic Projection
maps circles on $\mathbb{S}^2$ to circles and lines in the plane.

The case of lines is easier: The line in the image plane and the projection center define a plane. This plane intersects the sphere in the preimage circle of the line. – In other words: all circles on the sphere which pass through the projection center are mapped to lines.

For every circle $\{(\xi,\eta);(\xi-m)^2+(\eta-n)^2=r^2\}$ in the image plane one can easily compute the plane of the preimage circle, while the other direction needs a case distinction because of the lines.
We put $\xi^2+\eta^2 := 2m\xi+2n\eta+r^2-m^2-n^2$ into the third component of the preimage formula. Then clearly

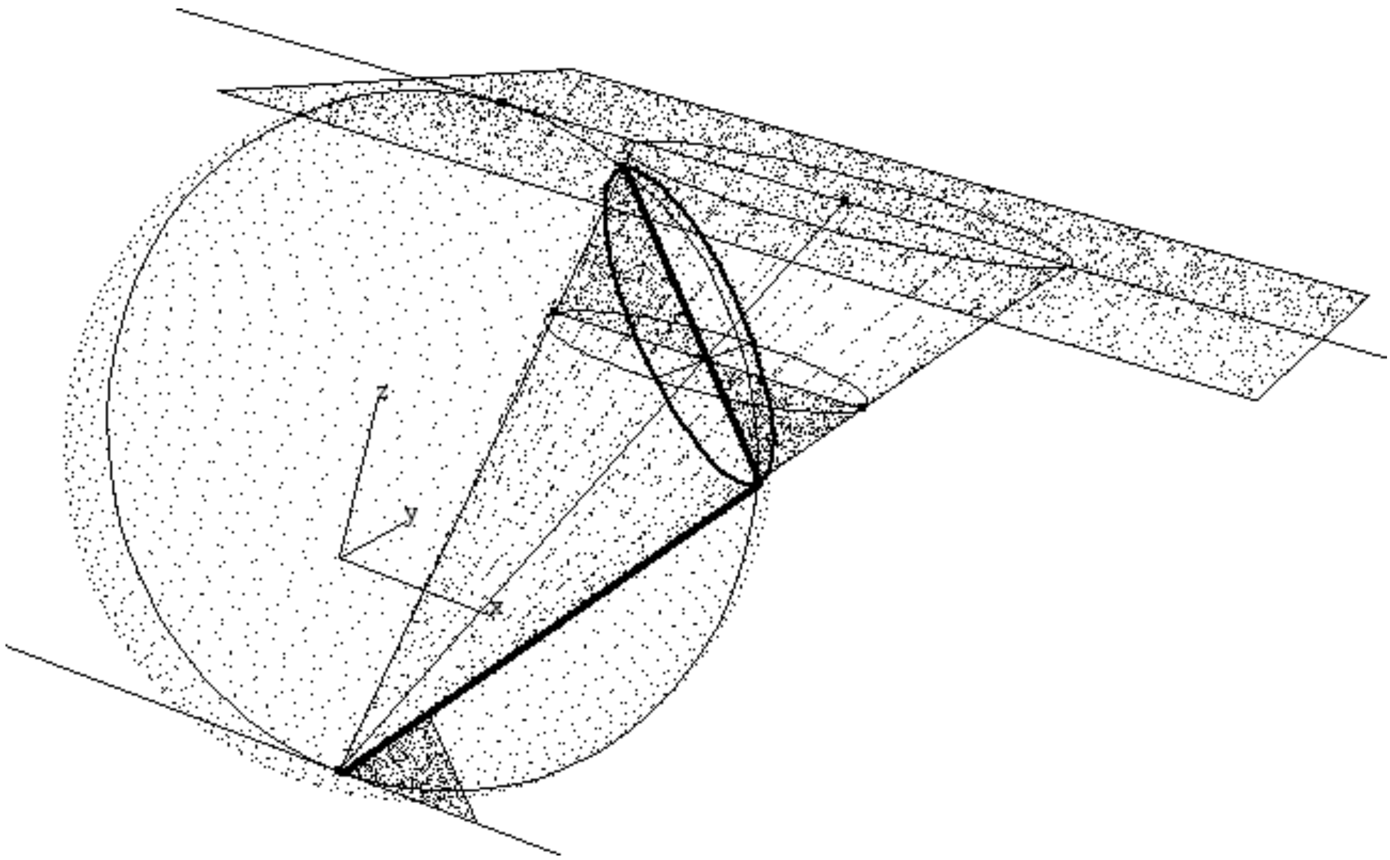
$$(1+\xi^2+\eta^2)\cdot(z+mx+ny+r^2-m^2-n^2-1)=0,$$

which shows the equation for the plane of the preimage.

Claim: Stereographic Projection preserves angles.

Through every point $p\in\mathbb{S}^2$ and tangential direction v there exists a circle on the sphere which passes through the projection center and through p and is tangential to v. Every such circle is mapped to a line which is **parallel** to the tangent of that circle at the projection center. The image lines of two such circles therefore intersect with the same angle as the two preimage circles, and the two preimage circles were chosen to represent an arbitrary angle on the sphere. – One can also prove this by computing with the derivative of the stereographic projection.

The image in 3D-XplorMath shows a nice proof of why circles are mapped to circles. A circle on the sphere and the projection center define a quadratic cone (unless the circle passes through the center and the image is a line). In general, therefore, the image of the circle is an ellipse. And all planes parallel to the image plane cut this cone in similar ellipses.



To prove that they are circles we show that their axes are equal. We add the line which joins the projection center and the midpoint of the image ellipse and we intersect it, at m, with the diameter of the circle (on the sphere) that is in the symmetry plane of the figure. Now consider the ellipse which intersects the cone in the parallel plane through m. Its axis orthogonal to the symmetry plane and the mentioned diameter of the circle are two intersecting secants of the circle. For their two subsegments holds: $\text{axisA}^2 = D1 \cdot D2$. The other, axisB, has its two halfs as edges of similar triangles which also have the edges $D1, D2$. So we conclude $D1 \cdot D2 = \text{axisB}^2$, hence $\text{axisA} = \text{axisB}$. – Note that the three shaded triangles are similar: two of them are bounded by parallel segments and the bottom triangle has the angle at the cone vertex equal to the angle of the top triangle opposite axisB.

A remarkle property of the formulas for stereographic projection and its inverse is: *They map points with rational coordinates to points with rational coordinates!* For example, if we stereographically project a tangent line to the unit circle, then the rational points on the line give us all rational points on the circle! Multiply $(p/q)^2 + (s/t)^2 = 1$ by the denominators to get all Pythagorean triples.

A differential geometric result in Dimension 3 and higher says: Any angle preserving map between spaces of constant curvature maps planes and spheres to planes and spheres. For people with that background it is therefore obvious that stereographic projection maps circles to circles.

In 3D-XplorMath one has stereo vision available so that one can see visualizations of images on the Riemann sphere in 3D. This shows the symmetries, for example, of the doubly periodic functions much more clearly than their images in the Gaussian plane do.

HK.