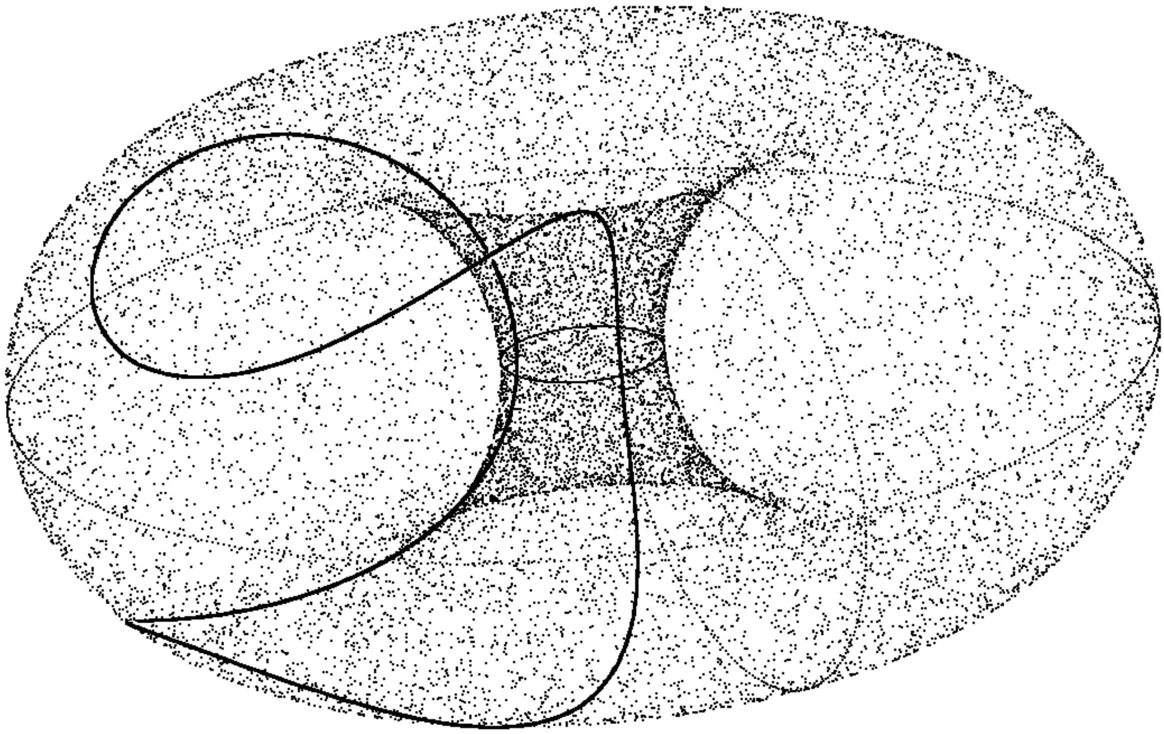
Space Curves of Constant Curvature on Tori*

These are the simplest non-planar closed constant curvature space curves that we have so far met. Their existence proof depends only on symmetry arguments. Example:



The program 3D-XplorMath allows to switch (in the Action Menu) between such curves on three surface families with rotation symmetry and equator mirror symmetry: namely on tori, ellipsoids and cylinders. The meridian curves of the tori and ellipsoids and the cross sections of the horizontal cylinders are ellipses with vertical axis cc and horizontal axis bb. The midpoints of these ellipses, in the torus case, lie on a circle of radius $aa\,(> bb)$. The

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

cylinders, viewed as limits of the tori, have their lengths controlled by aa and the rotation symmetry degenerated to translation symmetry. The ellipsoids are described as tori with $aa = 0$.

Space curves of constant curvature $k = dd$, which lie on a given surface, can easily be computed via the ODE below, if the desired space curvature is chosen *larger* than all the normal curvatures of the surface. (In fact, only those normal curvatures which the curve meets, do matter.) We are interested in curves which are symmetric with respect to the equator plane and with respect to some meridian plane - such curves are made up of four congruent arcs and are automatically closed. The initial point is therefore chosen on the equator and the initial direction is vertical. The ODE is such that the angle, with which the meridians are intersected, increases until $90°$ is reached and we have obtained the quarter piece which gives our closed curve.
One can have the initial point on the inner or the outer equator of the torus by switching the sign of the parameter bb.
The angle between the initial direction and the vertical meridian can be set. It is $\pi * ee$. All these curves have selfintersections, but they give some feeling, how constant curvature space curves wind around on the given surface. With more care we can allow constant space curvature which is *smaller* than the maximum of the normal curvatures of the surface. Below we will find closed ones also among these. – First we turn to the ODE.

The ODE for Constant Space Curvature

For every tangential unit vector $\vec{v}$ surfaces have a normal curvature $b(\vec{v}, \vec{v})$, where $b(\ ,\ )$ is the second fundamental form of the surface. Here we describe surfaces as the levels of a function $f : \mathbb{R}^3 \mapsto \mathbb{R}$, where the 'level' is the set of points where the function f has the value 0. (This constant can be changed with the parameter ff.) This description as a level of f allows to compute the *normal curvature* as

$$\kappa_n(\vec{v}) = \langle d_{\vec{v}} \operatorname{grad} f,\ \vec{v} \rangle / |\operatorname{grad} f|.$$

A curve on the surface with tangent vector $\vec{v}$ will have space curvature dd at that point if the *tangential curvature* (also called *geodesic curvature*) is

$$\kappa_g(\vec{v}) = \sqrt{dd^2 - \kappa_n(\vec{v})^2}.$$

Note that $N := \operatorname{grad} f / |\operatorname{grad} f|$ is the prefered unit normal field of the torus. The desired curve on the surface is therefore determined by the ODE:

$$c''(s) = \kappa_n(c'(s)) \cdot N(c(s)) + \kappa_g(c'(s)) \cdot c'(s) \times N(c(s)).$$

Any solution of this ODE with

$$f(c(0)) = 0 \text{ and } c'(0) \perp \operatorname{grad} f(c(0))$$

stays on the level $\{f = 0\}$, i.e. on the given surface, and

is a space curve of constant curvature $k = dd$.

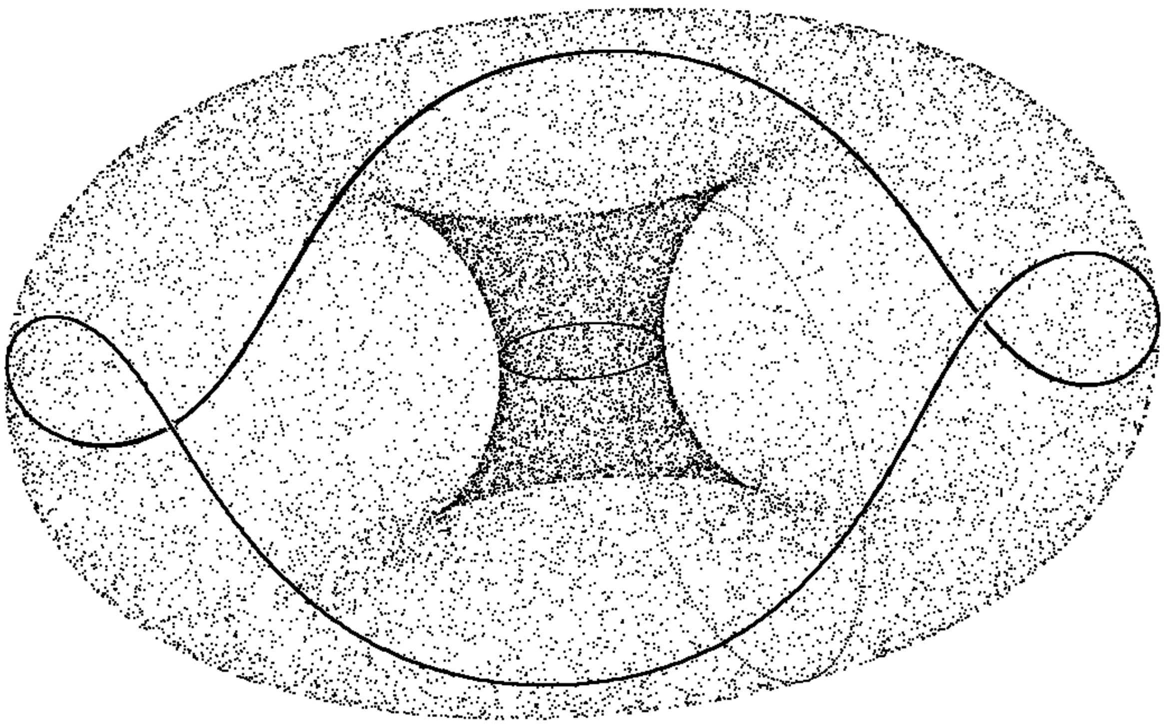
To force such curves, with simple arguments, to close up we need to employ symmetries of the surface. Therefore we use this ODE on surfaces of revolution which, in addition, have a reflection symmetry orthogonal to the axis of rotation. The previous argument works on such surfaces.

More Closed Constant Curvature Curves.

The idea is to look for other symmetries of the curves. Our surfaces allow 180° rotations around normals at any equator point. Such symmetries rotate an arc of positive geodesic curvature κ_g into an arc with $\kappa_g < 0$, i.e. we need $\kappa_g = 0$ where the curve crosses the equator. The only choice for the space curvature therefore is:

$$k = dd := \pm\kappa_n(c'(0)), \text{ hence } \kappa_g := \pm\sqrt{dd^2 - \kappa_n^2}.$$

We choose $\kappa_g > 0$ above the equator, $\kappa_g < 0$ below it. Note that on the cylinder there are helices with these initial conditions. They solve our ODE. On the other hand, on the torus and on the ellipsoid of revolution the latitudes have smaller radius than the equator. Angular momentum conservation therefore requires that the solution curves increase their angle against the meridians. Exactly as in the simpler case above they turn until they intersect a meridian orthogonally and then continue reflection symmetric (with respect to the plane of the meridian). This symmetry implies that they reach the equator again when their geodesic curvature is zero. Therefore they can cross the equator as smooth curves and the continuation agrees with the 180° normal rotation symmetry! And so on at all further crossings until the solution comes around the surface and to the vicinity of the initial point. In general it will not close. We can vary the size of the equator (*aa* for the torus, *bb* for the ellipsoid) until the solution hits the initial point. There, it is either half a period off or, because of the angular momentum, it reaches the initial point with the same tangent, as a smoothly closed curve! This constructs many closed constant curvature space curves, because we have the parameters aa, bb, cc, ee to play with. – Numerically we can use the morphing feature of 3D-XplorMath and appeal to the intermediate value theorem to find solutions.



Surprisingly, we can find these oscillating curves also on circular cylinders. We start the integration where we expect the reflection symmetry: tangential to a straight line and with geodesic curvature $\kappa_g(c(0)) \in (0, \max \kappa_n)$. For the solution the angle against the straight lines will increase and the geodesic curvature decrease until it becomes zero. If we call the straight line through that point *equator of the cylinder*, then we have on the cylinder the same kind of curve that we obtained before on tori and ellipsoids.
In the Action Menu of 3D-XplorMath one can switch between the described two kinds of symmetries of the curves.

Some Numerical Remarks

The helices on the cylinders show that we should expect trouble when we try to solve our ODE numerically with initial value $\kappa_g(c(0)) = 0$. Recall that the Runge-Kutta method needs to make *four first order trial steps* before the high accuracy Runge-Kutta step is obtained as an average of those four trial steps. These trial steps cannot always be computed because $dd^2 - \kappa_n^2 < 0$ at the endpoint of some trial step. Currently I do not know $c'''(0)$ for the theoretically constructed curves (the even derivatives vanish because of the 180° symmetry). Therefore I cannot construct a numerical method which avoids the above problem.

Instead I solve a slightly wrong equation by defining a slightly too large space curvature:

$$k = dd := \left|\kappa_n(c(0), c'(0))\right| + 0.00001.$$

This error is big enough for Runge-Kutta to proceed and small enough so that the osculating circles, while drawing the evolute, show no discontinuity. (The evolute increases errors of the curve very much.)

Warning: If the selection in the Action Menu is such that the curves with two orthogonal reflection symmetries are computed then the user may set the space curvature $k = dd$ *arbitrarily*. To avoid crashes caused by square roots of negative numbers the program computes $\sqrt{\max(0, dd^2 - \kappa_n^2)}$. The computed curves in such situations are geodesics, not curves with constant space curvature.

H.K.