About Anand-Ward Solitons*

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Generally speaking, solitons are solutions to nonlinear wave equations that exhibit particle-like behaviour, but the term is frequently retricted further by demanding that the wave equation be "integrable" in one of several technical senses of that term. The earliest known soliton equations (Korteweg-DeVries, Sine-Gordon, Cubic Schroedinger, ...) were all in one space dimension, and for a time it was even suspected that soliton-like behavior could not occur in higher dimensions.

The solutions in this subcategory are all pure soliton solutions of a modified Chiral Model introduced by Richard Ward. One 'reason' that this model is integrable is that it is a reduction of the Yang-Mills equations, or the Bogomolny equations in an indefinite signature. It is this relationship which allows the solutions to be constructed using twistor or inverse-scattering methods.

Twistor theory works by setting up a correspondence between analytic objects (e.g. solutions of differential equations) and certain geometric objects (e.g., complex analytic functions with prescribed poles, holomorphic vector bundles). In our case, the correspondence is between those solutions of Ward's model that extend analytically to the

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

compactification of space-time (a highly restrictive 'boundary condition at infinity') and holomorphic bundles on a two-dimensional, compact, complex manifold. In analogy with Liouville's theorem, stating that a bounded, holomorphic function on the complex plane is constant, Serre's GAGA principle states that certain objects on compact, complex spaces, known a priori only to be complex analytic, are necessarily algebraic. Think of this as an existence theory. The next step is to find an efficient way to represent the associated vector bundles. It turns out that there is a nice representation in terms of "monads" (which are matrices satisfying certain relations), and one can even write the solutions and energy density down explicitly—without any integrals or derivatives, in terms of the monad data. This is how the pre-programmed solutions here have been created, and it is how you can construct further solutions by entering monad data using the Setup AW solitons... dialogue.

To get solutions of Ward's equations, the matrices must satisfy the relation

$$\alpha_1 \cdot \gamma - \gamma \cdot \alpha_1 + b \cdot a = 0.$$

You are responsible for checking this equation yourself. There is also an additional nondegeneracy condition to insure that the associated solution is not singular, and a condition that corresponds to the solution being static, but the description of these conditions is too complicated to include here.

Under the Set Parameters... menu, you can also change the way the energy density is displayed. The parameters *bb* and *cc* scale distance and energy respectively; *aa* is time; and *dd* is a cut-off value for the energy above which the value *ee* will be plotted (this is because some solutions can have very tall "spikes" at certain times).

The full story, including the derivation of all these conditions and the geometry behind the construction can be found in my preprints at:

http://www.maths.warwick.ac.uk/~anand/preprints.html

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