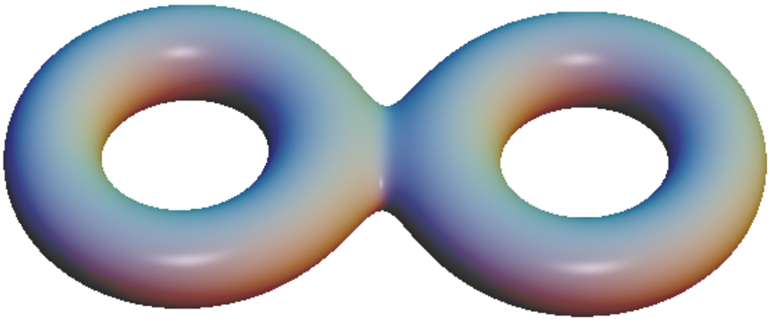
Explicit versus Implicit Surfaces,
in particular Level Sets of Functions *



Surfaces in $\mathbb{R}^3$ are either described as *parametrized images* $F : D^2 \to \mathbb{R}^3$ or as *implicit surfaces*, i.e., as levels of functions $f : \mathbb{R}^3 \to \mathbb{R}$, as the set of points where f has some given value, i.e. $\{x \in \mathbb{R}^3;\ f(x) = given\}$. *Graphs* of functions $h : \mathbb{R}^2 \mapsto \mathbb{R}$ are both: $F(u, v) := (u, v, h(u, v))$ is a *parametrization* and $f(x, y, z) := h(x, y) - z$ is a level function, $f = 0$ the *implicit equation*.
For most simple surfaces one has both representations, examples in 3DXM: *All Quadratic Surfaces, Tori, Cyclides, Cross-Cap, Steiner Surface, Algebraic Boy Surface, Whitney Umbrella.* In each case, the explicit and the implicit version open the same `Documentation`.
One can more easily make images of parametrized surfaces than of implicit surfaces, because every point $p \in D$ can be mapped with the given function F to obtain 'explicitly' a point $F(p)$ of the surface. Note however that the opposite problem: "Given a point in $\mathbb{R}^3$, decide whether it lies on the surface" does not have an easy answer. For an implicit

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

surface, on the other hand, it is easy to decide whether a given point in $\mathbb{R}^3$ is on the surface (simply check $f(x)$), but no point is given explicitly, one has to use some algorithm to find points $x \in \mathbb{R}^3$ which satisfy $f(x) = \textit{given value}$. Even after one has found many points on the surface, how does one connect them, what is a good way to represent the surface? The method of *raytracing* has been invented as one solution. Choose some center point C, think of it to be near the eyes of the viewer. Connect each pixel of the screen with C by a line and decide whether this line meets the surface. If it does then of all the intersection points on the line choose the one closest to C, compute the normal of the surface at this point x (i.e. compute grad $f(x)$) and decide with this information what light (from fixed light sources) will be reflected by the surface at x towards C. Color that pixel accordingly. In this way one produces an image which presents the surface as if it were an illuminated object. The computation used to take very long, but todays computers do such pictures while you wait, but not quite fast enough for real time rotations. These pictures look very realistic, but of course they show only what is visible from the viewer, in particular: farther away parts of the surface can be hidden by nearer parts.
In 3D-XplorMath a second method is offered. Imagine that the surface is intersected with random lines until around 10 000 points have been found on the implicit surface. Then red-green stereo is used to project these points to the screen. When viewing through stereo glasses we see all these points

in the correct position in space and our brain interpolates them and lets us see a surface in space. This representation shows all parts of the surface (within some viewing sphere), not just the front most portions. Since one can achieve fairly uniform distributions of points on level surfaces, one sees many points in the direction towards *contours* of the surface. This emphasis of the contour points is so strong that one gets a fair impression of the surface even if one does not look through red-green glases. This method is fast enough for real time rotations.

Once an implicit surface has been drawn, one has solved the problem of computing the 3D-data of surface points selected by mouse on the screen. One can therefore more easily move geometric attributes, like curvature circles, around on an implicit surface than on a parametrized surface. See in both cases the Action Menu entry `Move Principal Curvature Circles`.

What surfaces can one see?

In addition to the simple surfaces already mentioned we have two groups. *Alebraic Surfaces* which have been studied because of their *singularities*, these have established names and extensive literature. And *Compact Surfaces of higher genus*, these are added because such surfaces do not come with explicit parametrizations. (Their names are given in 3DXM and not known elsewhere.) Already fairly simple functions may have level surfaces which are more complicated than tori, they are called bretzel surfaces of genus $g > 1$.

How to find functions with compact levels of genus ≥ 2. As an example, consider *two circles* of radius $r = 1$, in the x-y-plane, with midpoints $\pm cc$ on the x-axis. These two circles are described as the intersection of the x-y-plane: $\{g(x, y, z) := z = 0\}$ with the zero set of the function

$$h(x, y, z) := \frac{((x - cc)^2 + y^2 - 1) \cdot ((x + cc)^2 + y^2 - 1)}{1 + (1 + cc)(x^2 + y^2)}.$$

The denominator prevents the function from growing too fast, the weight factor $1 + cc$ is experimental. Next define

$$f(x, y, z) := h(x, y, z)^2 + (1 + cc)g(x, y, z)^2.$$

Clearly, the zero set of f is the union of the two circles, which is *not* a surface, because $\operatorname{grad} f$ vanishes along this zero set. However, most of the levels $\{(x, y, z); f(x, y, z) = v > 0\}$ are surfaces without singularities. If the two circles intersect $(0 < cc < 1)$, then for small $v = ff$ the levels are the boundary of a thickening of the two circles, i.e., surfaces of genus 3. As ff increases either the middle hole or the two outside holes close first (depending on cc). For large ff the level surfaces are (not completely round) spheres. Each time such a topological change occurs we observe one special surface, it is not smooth like the other levels, but has one or more cone like singularities.
If $cc > 1$ then, for small ff, the levels are disjoint tori. As ff increases, either the tori grow together to a genus 2 surface, or the holes of the tori close first and later the two sphere-like surfaces grow together.

This family is called *Pretzel* in 3DXM.

Functions with compact levels in 3D-XplorMath

One should always experiment with the level value v of the function f. In 3DXM: $v = ff$. For small values of ff one will see how the function was designed by guessing the degenerate level $f = 0$. The `Default Morph` often varies ff, for example showing non-singular levels converging to the singular one. In some cases other parameters are morphed, for example to get larger values of the genus g. Some cases offer: `Flow to Minimum Set` $\{f = 0\}$ (see Action Menu). (Artificial looking denominators in the following prevent the function f from growing too fast.)
Note that the Action Menu has many decorations for implicit surfaces: Curvature line fields, net of curvature lines, normal curvature circles, geodesics with mouse chosen initial data, geodesic nets.

Pretzel: See page 5 of *Explicit versus Implicit Surfaces.* The surface has genus 0,1,2 or 3, depending on parameter values.
$f(x, y, z) := h(x, y, z)^2 + (1 + cc)z^2$ with

$$h(x, y, z) := \frac{((x - cc)^2 + y^2 - 1) \cdot ((x + cc)^2 + y^2 - 1)))}{1 + (1 + cc)(x^2 + y^2)}$$

Bretzel2, a genus 2 tube around a figure 8, genus 0 for large ff:

$$f(x, y, z) := \frac{(((1 - x^2)x^2 - y^2)^2 + z^2/2)}{(1 + bb(x^2 + y^2 + z^2))}.$$

Bretzel5, a genus 5 tube around two intersecting ellipses:
$f(x, y, z) := ((x^2 + y^2/4 - 1) \cdot (x^2/4 + y^2 - 1))^2 + z^2/2.$

Pilz, a genus 3 tube around circle and orthogonal ellipse:
$f(x, y, z) :=$
$((x^2+y^2-1)^2+(z-0.5)^2)\cdot(y^2/aa^2+(z+cc)^2-1)^2+x^2)$
$-\,dd^2(1+bb(z-0.5)^2)$.
`Default Morph:`$0.03 \leq cc \leq 0.83$.

Orthocircles, a genus 5 tube around three intersecting orthogonal circles ($aa = 1,\ ff = 0.05$) or a tube around three Borromean ellipses ($aa = 2.3, ff = 0.2$) – choose in the Action Menu.
$f(x, y, z) :=$
$((x^2/aa + y^2 - 1)^2 + z^2)\cdot((y^2/aa + z^2 - 1)^2 + x^2)\cdot$
$((z^2/aa + x^2 - 1)^2 + y^2)$.
Use: `Flow to Minimum Set` $\{f = 0\}$ (from Action Menu).

DecoCube, tube around six circles of radius cc on the faces of a cube. Genus 5,13,17, depending on cc, ff:
$f(x, y, z) := ((x^2 + y^2 - cc^2)^2 + (z^2 - 1)^2)\cdot$
$((y^2+z^2-cc^2)^2+(x^2-1)^2)\cdot((z^2+x^2-cc^2)^2+(y^2-1)^2)$.
`Default Morph:` $ff = 0.02,\ 0.25 \leq cc \leq 1.3$.
Use: `Flow to Minimum Set` $\{f = 0\}$ (from Action Menu).

DecoTetrahedron has as its minimum set four circles on the faces of a tetrahedron. The formula is similar but more complicated than the previous one. cc changes the radius of the circles, bb changes their distance from the origin, ff selects the level. Use: `Flow to Minimum Set` to see the circles used for the current image.
The `Default Morph` changes cc and with it the genus.

JoinTwoTori is a genus 2 surface such that the connection between the two tori does not much distort them if ff is small. (It is used for genus-2-knots in Space Curves.)

$$\begin{aligned} Tor_{right} &:= ((x-cc)^2+y^2+z^2-aa^2-bb^2)^2 \\ &\qquad +4aa^2(z^2-bb^2) \\ Tor_{left} &:= ((x+cc)^2+y^2+z^2-aa^2-bb^2)^2 \\ &\qquad +4aa^2(z^2-bb^2) \end{aligned}$$

$$f(x,y,z) := \frac{Tor_{right}\cdot Tor_{left}}{1+(x-cc)^2+(x+cc)^2+y^2+z^2/2}.$$

The `Default Morph:` $0.01 \le ff \le 2.5$ joins the tori.

CubeOctahedron

The level surfaces of the function

$f_{cube}(x,y,z) := \max(|x|,|y|,|z|)$ are cubes.

The level surfaces of the function

$f_{octa}(x,y,z) := |x|+|y|+|z|$ are octahedra.

$\tilde{a} := \min(2\cdot aa, 1)$, $\tilde{b} := 2\cdot\min(bb,1)$. These coefficients for the following linear combination allow an interesting morph.

$$f(x,y,z) := \max(\tilde{a}\cdot f_{octa}(x,y,z)+\tilde{b}\cdot f_{cube}(x,y,z)).$$

Default: $aa = 0.5, bb = 1, ff = 1$. This truncated cube is Archimedes' Cubeoctahedron.

`Default Morph:` $aa = \frac{2}{3} \to \frac{1}{3}$, $bb = 0.5 \to 1.5$, $ff = 1$. This deformation from the octahedron to the cube passes through three Archimedean solids.

Algebraic Functions with Singularities in 3DXM

CayleyCubic :

$f(x, y, z) := 4(x^2 + y^2 + z^2) + 16\, x\, y\, z - 1, \quad ff = 0.$

This cubic has 4 cone singularities at the vertices of a tetrahedron. The other surfaces in the `Default` ff`-Morph` are nonsingular.

ClebschCubic :

$f(x, y, z) :=$
$81(x^3 + y^3 + z^3) - 189(x^2(y+z) + y^2(z+x) + z^2(x+y)) + 54xyz + 126(xy + yz + zx) - 9(x^2 + x + y^2 + y + z^2 + z) + 1.$

This cubic has no singularities but is famous for the 27 lines that lie on it. The lines are shown in 3DXM. The surface has tetrahedral symmetry.

DoublyPinchedCubic :

$f(x, y, z) := z(x^2 + y^2) - x^2 + y^2.$

This cubic has two pinch-point singularities at ± 1 on the z-axis. The segment between the singularities lies on it. The whole z-axis satisfies the equation; the `Default Morph` shows how an infinite spike converges to this line.

KummerQuartic :

$\lambda := (3aa^2 - 1)/(3 - aa^2),$

$$f(x, y, z) := (x^2 + y^2 + z^2 - aa^2)^2 - \lambda((1-z)^2 - 2x^2)((1+z)^2 - 2y^2), \quad aa = 1.3.$$

This quartic has 4+12 cone singularities and tetrahedral symmetry. Six noncompact pieces, each with two cone points, are connected by five compact pieces which look like curved tetrahedra. The singularities survive small changes, see the `Default Morph` : $1.05 \le aa \le 1.5, ff = 0.$

BarthSextic :
$c_1 := (3+\sqrt{5})/2,\ c_2 := 2+\sqrt{5}$
$f(x,y,z) :=$
$4(c_1x^2-y^2)(c_1y^2-z^2)(c_1z^2-x^2)-c_2(x^2+y^2+z^2-1)^2.$
Barth's Sextic has icosahedral symmetry. 20 tetrahedron-like compact pieces are placed over the vertices of a dodecahedron so that each tetrahedron has 3 of its vertices at midpoints of dodecahedron edges. This accounts for 30 of the cone singularities. Each of the 20 outward pointing vertices of the tetrahedra is connected via a cone singularity to a cone-like noncompact piece of the Sextic. The `Default Morph` embeds this singular surface in a family of nonsingular sextics. `Use Raytrace Rendering`.

D4 :
$f(x,y,z) := 4x^3+(aa-3x)(x^2+y^2)+bbz^2$
This family of cubics has a $D4$-singularity. At $bb=0$ the family degenerates into three planes, intersecting along the z-axis.

UserDefined: Our example is the *Cayley Cubic*, see above.

H.K.