The Korteweg-de Vries Equation*

NOTE: For a fuller understanding of the following, you may find it helpful to first read "About This Category" from the Documentation Menu and "Introduction to Wave Equations" from its Topics submenu.

The partial differential equation,

$$u_t(x,t) + u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0$$

for a real-valued function, u, of two real variables x and t (space and time) is known as the Korteweg-de Vries Equation (or simply KdV). It was first derived in 1895 by D.J. Korteweg and G. de Vries to model water waves in a shallow canal. Their goal was to settle a long-standing question; namely whether a solitary wave could persist under those conditions. Based on personal observations of such waves in the 1830's, the naturalist John Scott Russell insisted that such waves do occur, but several prominent mathematicians, including Stokes, were convinced they were impossible.

Korteweg and de Vries proved Russell was correct by finding explicit, closed-form, traveling-wave solutions to their equation that moreover decay rapidly spatially, and so represent a highly localized moving lump. In fact, they found a one-parameter family of such solutions:

$$u(x,t) = 2a^2 \operatorname{sech}^2(a(x-4a^2t), \quad \text{(in 3DXM: a = cc)}$$

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

and these are the traveling wave solutions whose evolutions are displayed by 3D-XplorMath, with its parameter cc playing the rôle of the constant a. Note that the wave's velocity, $4a^2$, is proportional to its amplitude, $2a^2$, meaning that the taller waves of this family move faster. Both the fact that such a solution to a non-linear equation could exist and that one could write it in explicit form were later recognized to be highly important, although originally these facts were relatively unnoticed.

The KdV equation did not receive significant further attention until 1965, when N. Zabusky and M. Kruskal published results of their numerical experimentation with the equation. Their computer generated approximate solutions to the KdV equation indicated that any localized initial profile, when allowed to develop according to KdV, asymptotically in time evolved into a finite set of localized traveling waves of the same shape as the original solitary waves discovered in 1895. Furthermore, when two of the localized disturbances collided, they would emerge from the collision as another pair of traveling waves with a shift in phase as the only consequence of their interaction *. Since the "solitary waves" which made up these solutions seemed to behave like particles, Zabusky and Kruskal coined the name "soliton" to describe them. The formula

$$u(x,t) = \frac{12\ gg\ (3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{\sqrt{(3\cosh(x - 28t) + \cosh(3x - 36t))}}$$

* To see this phase shift, choose Display as Graph from the Action Menu.

is one of these solutions, consisting of two lumps and therefor called two-soliton solution of KdV, and it is one of the KdV solutions whose evolution can be displayed using 3D-XplorMath. Shortly after that, another remarkable discovery was made concerning KdV; a paper by C. Gardner, J. Greene, M. Kruskal, and R. Miura demonstrated that it was possible to derive many exact solutions to the equation by using ideas from scattering theory. In particular, the solutions whose evolution are shown by 3D-XplorMath are exact solutions that can be found by this method.

Using modern terminology, we could say that these authors showed that KdV was an *integrable* non-linear partial differential equation. It was the first to be discovered, but since then, many other such equations have been found to be integrable and admit soliton solutions, in particular the Sine-Gordon Equation (SGE) and the Nonlinear Schrödinger Equation (NLS)—solutions to both of which are also shown by 3D-XplorMath. However, KdV is usually considered the canonical example, in part because it was the first equation known to have these properties.

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