Forced Duffing Oscillator *

What is it?

What we shall call the *Forced Duffing Oscillator Equation* is the second order ODE for a single variable x,

$$\frac{d^2x}{dt^2} = -hh\,x - ii\,x^3 - aa\,\frac{dx}{dt} + bb\,\cos(cc\,t) \qquad (1)$$

whose solutions we display via the equivalent (non-autonomous) first order system in two variables, x and y:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -hh\,x - ii\,x^3 - aa\,y + bb\,\cos(cc\,t) \qquad (2)$$

which in turn can be made into an autonomous first order system in three variables, T, x and y:

$$\begin{aligned} \frac{dT}{dt} &= 1, \ \frac{dx}{dt} = y, \\ \frac{dy}{dt} &= -hh\,x - ii\,x^3 - aa\,y + bb\,\cos(cc\,T). \end{aligned} \qquad (3)$$

We discuss the interpretation and significance of the five parameters, aa, bb, cc, hh, ii below. Their default values are: $aa = 0.25, bb = 0.3, cc = 1.0, hh = -1.0,$ and $ii = 1.0$. If $bb{\cdot}cc \neq 0$ then the forcing period $2\pi/cc$ is shown by yellow dots on the orbit.

Why is it interesting?

Here are two of the considerations that make the oscillator equation (1) worth studying. First, with appropriate choices of parameter values it reduces to a variety of

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/ ODEs

mathematically and physically interesting oscillator models; some classical such as the harmonic oscillator (with and without damping and forcing) and others that are more exotic, such as the classic Duffing oscillator introduced by Duffing in 1918. By putting these together in a parametric family, we can investigate how various features of these systems behave as we move around in the parameter space. Secondly—and more importantly—it was in in the study of the Duffing Oscillator that symptoms of the phenomena we now call "chaos" and "strange attractor" were first glimpsed (although their significance was only appreciated later). By the Poincaré-Bendixson Theorem, three is the smallest dimension in which an autonomous system can exhibit chaotic behavior, and the Duffing system is so simple that it lends itself very easily to the study and visualization of the phenomena related to chaos.

The Newtonian Particle Interpretation.

Note that (1) becomes Newton's equation of motion for a particle of unit mass moving on the x-axis if we define the "force", $F(x, \frac{dx}{dt}, t)$, acting on the particle to be the right-hand side of (1). Let's interpret the various terms of F from this point of view.

If hh is positive then the term $-hh\,x$ by itself gives Hooke's Law for a spring, that "stress is proportional to strain" and the parameter hh has the interpretation of Hooke's proportionality factor between the extension of the spring, x, and the restoring force. If also $ii = 0$ then we have a pure Hooke's Law force that gives the Harmonic Oscillator, $\frac{d^2x}{dt^2} = -hh\,x$. But a real spring only satisfies Hooke's Law approximately, and the term $-ii\,x^3$ represents the next term in the Taylor expansion of the restoring force under the reasonable assumption that this force is an odd function of the spring extension, x. (If ii is positive it is called a "hardening" spring and if negative a "softening" spring.) For the classic Duffing Oscillator, hh is negative and ii is positive and there is not a good interpretation of the force in terms of a spring. Rather, the sum of the two terms $-hh\,x - ii\,x^3$ should be interpreted as the force on a particle that is moving in a double-well potential as we will discuss in more detail below.

The term $-aa\,\frac{dx}{dt}$ represents a "friction" force of the sort that would be experienced by a particle like a bullet traveling through air or a bead sliding on a wire; that is, assuming that the "damping" or "friction" coefficient aa is positive, it describes a force acting on the particle in the direction opposite to the velocity and with a magnitude that is proportional to the magnitude of the velocity.

Under the sum of the above terms of the force law F, the particle will (in general) oscillate back and forth—which of course is why it is called an oscillator—however if $aa > 0$ these oscillations will gradually die down as the kinetic energy is absorbed by friction. The final term in the force law, $bb\,\cos(cc\,t)$ is a periodic forcing term that will act on and perturb the motion of this oscillating particle, and we note that it is solely a function of the time and is independent of both the position and velocity of the particle. We will discuss a possible physical interpretation of this term later. The parameter bb is clearly the amplitude of this forcing term, i.e., its maximum magnitude, and the parameter cc is the angular velocity of its phase in radians per unit time, so that the period of the forcing term is $\frac{2\pi}{cc}$ and its frequency is $\frac{cc}{2\pi}$. As we shall see, it is the energy that is fed into the system by this forcing term that is essential for the interesting chaos related effects to occur. In fact the most interesting behaviors of solutions of (1) are present when all the above terms are present in F, that is when the oscillator is both forced and damped, and in fact the way damping and forcing can balance each other is crucial to understanding the general behavior of solutions. However we will begin by analyzing the simpler situation when both the damping and forcing terms are missing.

The Undamped, Unforced Case.

We now assume that aa and bb are both zero, so the force $F(x) = -hh\,x - ii\,x^3$ is a function of x alone. Now in one-dimension, whenever this is the case the force is *conservative*, that is, it is minus the derivative of a "potential" function, $U(x)$. Indeed, if we define $U(x) := -\int_0^x F(\xi)\,d\xi$, then clearly $F(x) = -U'(x)$. If as above we write $y := \frac{dx}{dt}$, define the kinetic energy by $K(y) := \frac{1}{2}y^2$ and define the Hamiltonian or total energy function by $H(x,y) := K(y) + U(x)$, then $\frac{dH}{dt} = y\,\frac{dy}{dt} + U'(x)\,\frac{dx}{dt} =$

$y(\frac{dy}{dt} + U'(x))$. So, if Newton's Equation is satisfied, $\frac{dy}{dt} = \frac{d^2x}{dt^2} = F(x) = -U'(x)$, so $\frac{dH}{dt} = 0$. This of course is the law of conservation of energy: the total energy function $H(x, y)$ is constant along any solution of Newton's Equations. In one-dimension this provides at least in principle a way to solve Newton's Equation for any initial conditions $x = x_0$ and $y = y_0$ at time $t = t_0$. Namely, the path or orbit of the solution is a curve in the x-y plane, and by conservation of energy this curve is given by the implicit equation $H(x, y) = H(x_0, y_0)$. And since $\left(\frac{dx}{dt}\right)^2 = y^2 = 2K(y) = 2(H(x_0, y_0) - U(x))$, we find:

$$\frac{dt}{dx} = \frac{1}{\sqrt{2(H(x_0, y_0) - U(x))}},$$

so we can find t as a function of x by a quadrature, and then invert this relation to find x as a function of t.

In the Harmonic Oscillator case, with $hh = 1$ and $ii = 0$, $U(x) = \frac{1}{2}x^2$ so $H(x, y) = \frac{1}{2}(x^2 + y^2)$, so the orbits are circles, and it is easy to carry out the above quadrature and inversion explicitly, to obtain $x(t) = x_0 \cos(t - t_0) + y_0 \sin(t - t_0)$.

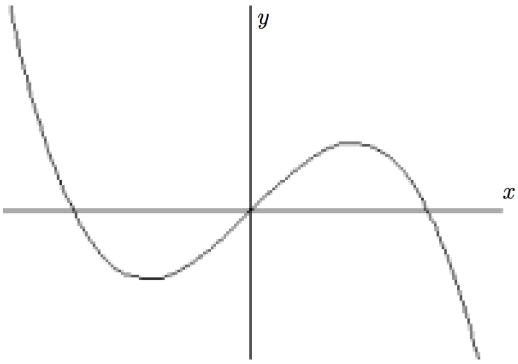
The Universal "Sliding Bead on a Wire" Model.

In one-dimension there is a highly intuitive physical model that makes it easy to visualize the motion of a particle under a given force. Moreover this model is "universal" in the sense that it works for all forces that are function of position only and hence, as we noted above, are of the form

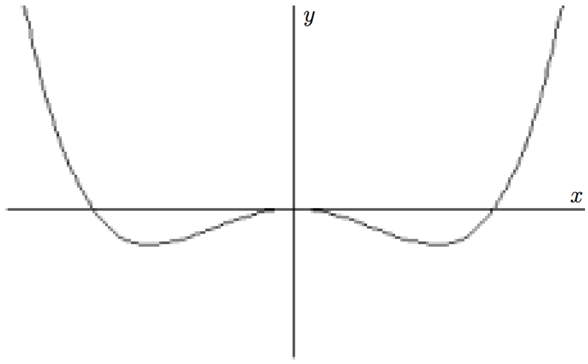
$F(x) = -U'(x)$ for some potential function U. Namely, imagine that we string a bead on a frictionless piece of wire that lies along the graph of the equation $y = U(x)$. If the bead has mass $m = 1$ and if we choose units so that g, the acceleration of gravity, equals one, then the gravitational potential of the bead is $mgh = h$ where h is its height. So if as usual we interpret the ordinate of a point as its height, then the gravitational potential of the bead when it is at the point $(x, y) = (x, U(x))$ is just $U(x)$, and the sliding motion of the bead along the wire under the attraction of gravity will exactly model whatever other system we started from!

In the case of the Harmonic Oscillator, where $F(x) = -x$ and $U(x) = \frac{1}{2}x^2$, the graph is the parabola, $y = \frac{1}{2}x^2$ and it is easy to imagine the bead oscillating back and forth along this parabola.

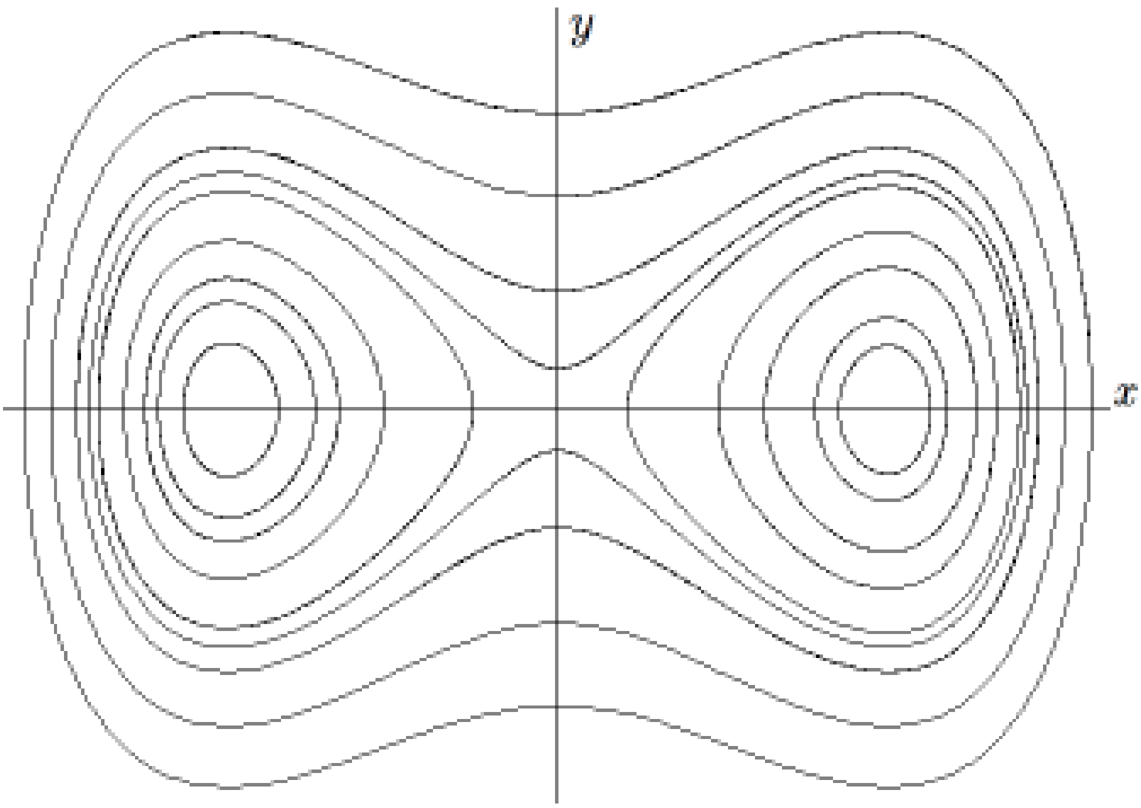
For the unforced and undamped Duffing Oscillator the force is $F(x) = -hh\,x - ii\,x^3$, where for simplicity in what follows we will assume that $ii > 0$ and $hh < 0$. The potential is $U(x) = \frac{hh}{2}x^2 + \frac{ii}{4}x^4$, which we note can be considered as the first two terms in the Taylor expansion for an arbitrary symmetric potential with local maximum at 0. It is easily checked that $\lim_{x\to\pm\infty} U(x) = +\infty$ and in addition to the local maximum at 0, there are two other critical points of U, at $x = \pm\sqrt{\frac{-hh}{ii}}$, where U has local minima. For the default values, $hh = -1$ and $ii = 1$, the force is $F(x) = x(1 - x^2)$, and the potential is $U(x) = \frac{1}{4}x^2(x^2 - 2)$, so the local minima are at ± 1. We graph this force $F(x)$ and potential $U(x)$ below, and show a selection of the resulting orbits. It should be clear why U is called a double-well potential.



$F(x)$



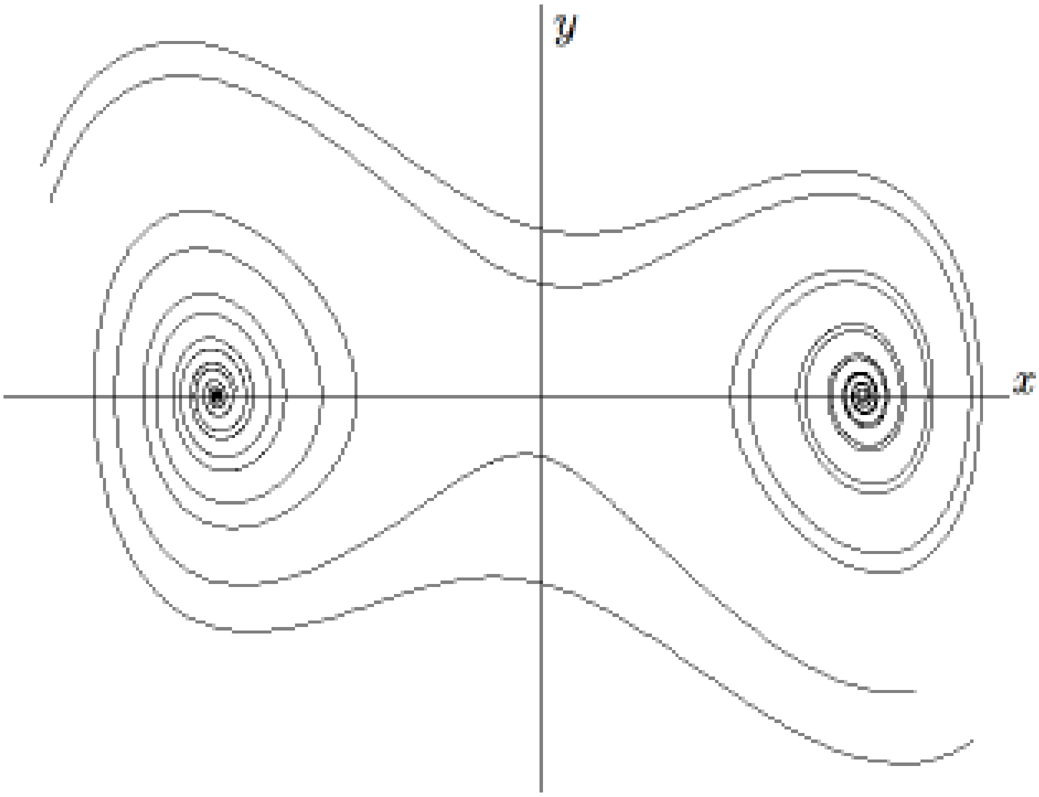
$U(x)$



Some orbits of the Unforced, Undamped Duffing Oscillator

The Unforced, Damped Duffing Oscillator.

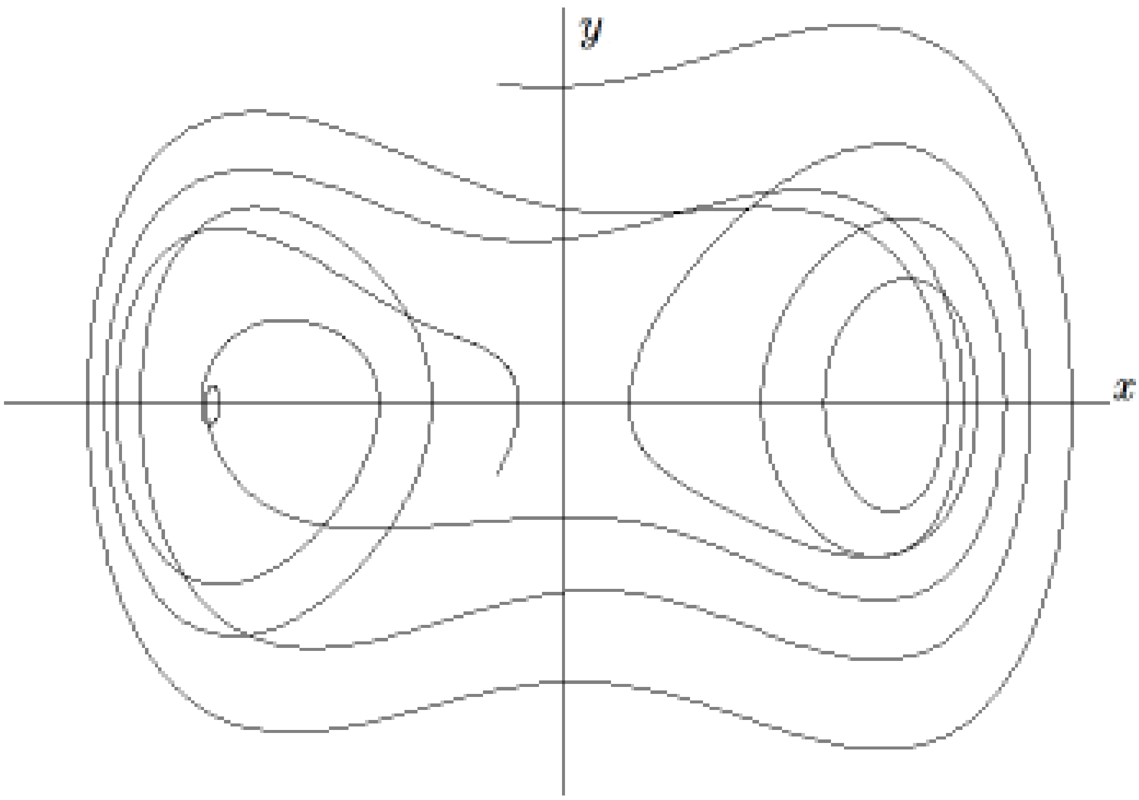
We now still assume $bb = 0$ (so there is no forcing) but we assume that $aa > 0$, so there is damping. In the bead on a wire picture, $aa\,\frac{dx}{dt} = aa\,y$ is the friction from the bead rubbing against the wire, and the force is now given by $F(x) = -U'(x) - aa\,y$. If we again calculate $\frac{dH}{dt}$ as we did above, we now find not $\frac{dH}{dt} = 0$ but instead $\frac{dH}{dt} = -aa\,y^2$. The result is that instead of the orbits of the bead in the x-y-plane being closed curves of constant total energy H, the energy decrease along the obits, and they cut across the $H =$ constant curves and spiral in towards the two minima of H at the bottom of the two potential wells. We show a selection of the resulting orbits below.



Some orbits of the Unforced, Damped Duffing Oscillator

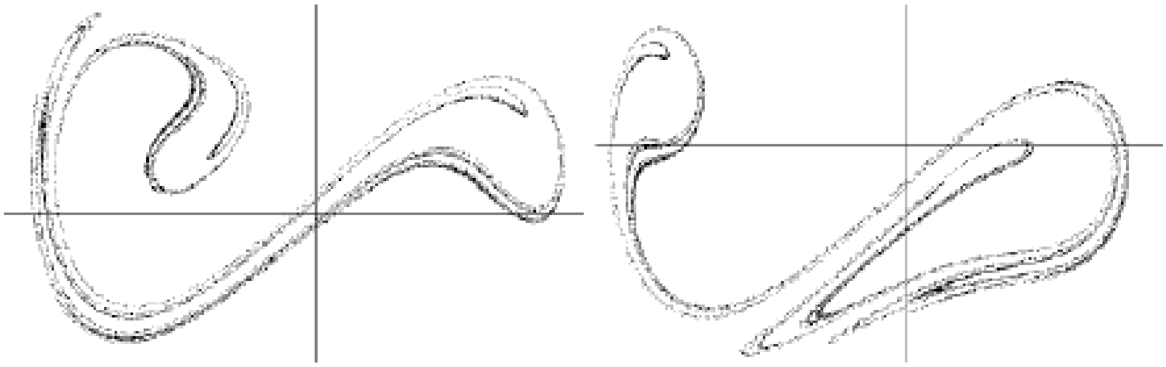
The Forced Duffing Oscillator.

We now add back the forcing term $bb \cos(cc\, t)$. First a word about how to interpret this force in the sliding bead picture. If we assume that there is an alternating electric field parallel to the x direction and with strength $\cos(cc\, t)$ at time t, then $bb \cos(cc\, t)$ will be the electric force felt by the bead if we give the bead an electric charge of magnitude bb.



Some orbits of the Forced, Damped Duffing Oscillator

Chaos, Strange Attractors, and Poincaré Sections.



Two Time Slices of the Duffing Attractor

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