

The Pseudosphere*

from a Sine-Gordon solution

The Pseudosphere was first found as a surface of revolution, with the Tractrix as meridian (see Planar Curves). It has Gauss curvature $K = -1$. See:

Constant Curvature Surfaces of Revolution.

Later in the 19th century it was discovered that surfaces with $K = -1$ can be constructed from soliton solutions of the Sine-Gordon Equation (SGE). This is explained in:

About Pseudospherical Surfaces,

which can be obtained from the Documentation Menu.

At about the same time, in 1868, Beltrami proved that the axiomatically constructed non-Euclidean geometry of Bolyai and Lobachevsky was the same as the simply connected 2-dimensional Riemannian geometry of Gauss curvature $K = -1$; for example the Riemannian metric of the Pseudosphere, extended to the plane: $du^2 + \exp(-2u)dv^2$. Their common name today is *Hyperbolic Geometry*.

The meridians are examples of *asymptotic geodesics*, a key notion in hyperbolic geometry. Curves, orthogonal to a family of asymptotic geodesics are called *horocycles* in hyperbolic geometry. They have infinite length in the simply connected case, on the Pseudosphere one sees finite por-

* This file is from the 3D-XplorMath project. Please see:

<http://3D-XplorMath.org/>

tions as the latitude circles.

In the theory which relates SGE solutions to surface in \mathbb{R}^3 of Gauss curvature $K = -1$, one first writes down the first and second fundamental forms in terms of such a solution $q(x, t)$ of SGE:

$$I = dx^2 + dt^2 + 2 \cos q \, dx \, dt, \quad II = 2 \sin q \, dx \, dt,$$

The Gauss-Codazzi integrability conditions are satisfied, because q is an SGE solution. One then obtains the first parameter line of the surface by integrating an ODE and the transversal other family by integrating a second ODE. The first and second fundamental forms above are written in asymptote coordinates, which means: the normal curvature of the surface in the direction of the parameter lines is 0. (Note that x and t are arc length parameters on the parameter lines. This leads to the Tchebycheff net mentioned in “About Pseudospherical Curves”.) Such parametrizations do not offer a good view of the surface. In 3DXM, therefore, the integration first creates one curvature line of the surface and secondly the orthogonal family of curvature lines with $u = x + t, v = x - t$. One can view this before the surface is shown with these parameter lines.

The SGE solution for the Pseudosphere is:

$$q(x, t) := 4 \arctan(\exp(x)).$$

H.K.