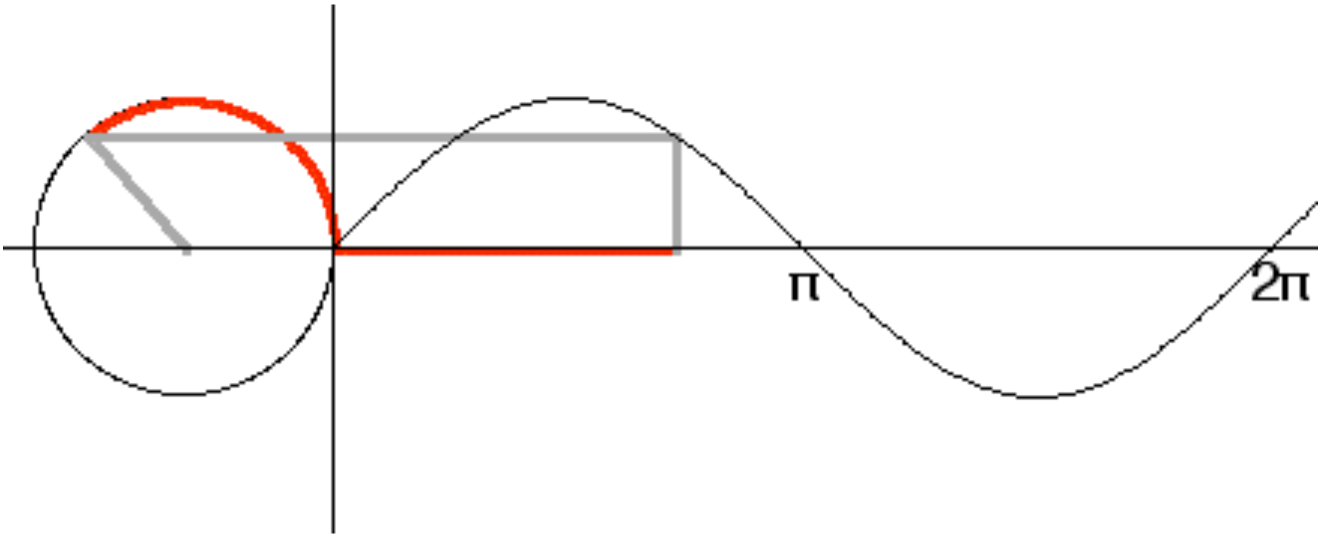
Sine *

The demo in 3D-XplorMath illustrates: If a unit circle in the plane is traversed with *constant velocity* then it is parametrized with the so-called trigonometric or circular functions,

$$c(t) = (\cos t, \sin t).$$



History

The earliest known computations with angles did not yet use degrees. In Babylonian astronomy angles were measured by their *chord*, i.e. by the base of an isocele triangle with the vertex in the center of a unit circle and the other vertices on the circumference. Hipparchus (180 - 125 BC) is credited with the first table giving chords in terms of degrees. The table proceeded in steps of $60°/16$. Ptolemy (ca 90 - ca 168 AD) computed more accurate tables in steps of $1°$. With these tables trigonometric functions had come

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

into existence, in modern notation the function

$$\text{chord: } \varphi \mapsto 2\sin\left(\tfrac{2\pi}{360}\tfrac{\varphi}{2}\right).$$

Astronomy needed computations with spherical triangles. Initially these were done in geometric form, based on theorems on quadrilaterals inscribed in circles. Around 800 AD Arab astronomers developed the more streamlined computations based on properties of the functions sin and cos, expressed in formulas similar to the ones in use today. Since Newton's time sin and cos are even more conveniently defined by their linear differential equations

$$\sin'' = -\sin,\ \sin(0) = 0,\ \sin'(0) = 1;$$
$$\cos'' = -\cos,\ \cos(0) = 1,\ \cos'(0) = 0.$$

The first derivatives $\sin' = \cos,\ \cos' = -\sin$ and addition theorems like

$$\sin(a+x) = \sin(a)\cos(x) + \cos(a)\sin(x),$$
$$\cos(a+x) = \cos(a)\cos(x) - \sin(a)\sin(x),$$

follow from the uniqueness theorems for such differential equations (both sides satisfy the same ODE and the same initial conditions). Also the identity

$$\sin^2(x) + \cos^2(x) = \sin^2(0) + \cos^2(0) = 1$$

which is needed to show "*that* sin *and* cos*, defined by their differential equations, indeed parametrize the unit circle with unit velocity*", follows directly by differentiation.

With Euler's discovery of the close connection between the exponential and the trigonometric functions computations became even more convenient, since one only needs

$$\exp' = \exp,\ \exp(0) = 1,\ \exp(z+w) = \exp(z)\exp(w)$$
$$\exp(x + i\,y) = \exp(x)\cdot(\cos(y) + i\sin(y)).$$

In the 19th century approximation of functions by *Fourier polynomials* $P_N(x) := a_0 + \sum_{n=1}^{N} a_n \cos(n\,x) + b_n \sin(n\,x)$ lead to unexpected difficulties and corresponding deep insights. The modern notion of a function and also the concept of continuity have their roots in these studies.

Today, when high school kids first meet sine and cosine, they do not meet them as real valued functions, but as ratios of edge lengths in rectangular triangles. There is no reference to a numerical algorithm that computes the values of sine and cosine from "angles". The above addition theorems emphasize this: if $(c_1, s_1), (c_2, s_2) \in \mathbb{S}^1$, then also $(c_1c_2 - s_1s_2, s_1c_2 + c_1s_2) \in \mathbb{S}^1$. Note that this addition of rational points gives again a rational point, and rational points on $\mathbb{S}^1$ are the same as Pythagorean triples, namely $(a/b)^2 + (c/d)^2 = 1$ is equivalent to $(ad)^2 + (bc)^2 = (bd)^2$. On this level sine and cosine belong to the geometry of similar triangles, and indeed, the addition formulas can be proved by using similarity of triangles. – So, what **is** the relation between sine, cosine and angles? One answer is to define sine, cosine in terms of their ODEs quoted above, another is, to discuss first the arc length of the circle, which means, define the inverse functions arcsine, arccosine first.

We come back to computations of sine and cosine below, but first we return to the beginning. The following formulas show (in modern notation) how the trigonometric functions enter planar and spherical geometry.

Laws of Sines and Cosines

For any *planar triangle* with side lengths a, b and c, whose opposite angles are α, β and γ respectively, we have:

Projection theorem: $c = a \cdot \cos\beta + b \cdot \cos\alpha$

Sine theorem: $b \cdot \sin\alpha = h_c = a \cdot \sin\beta$

Cosine theorem: $c^2 = a^2 + b^2 - 2ab\cos\gamma$

In astronomy the corresponding formulas for *spherical triangles* on the unit sphere played a much more important role. They are:

Projection theorem: $\cos a \sin c =$

$$\sin a \cos c \cos\beta + \sin b \cos\alpha$$

Sine theorem: $\sin b \cdot \sin\alpha = \sin a \cdot \sin\beta$

Cosine thm: $\cos c = \cos a \cos b + \sin a \sin b \cos\gamma.$

A consequence of the first two theorems is the

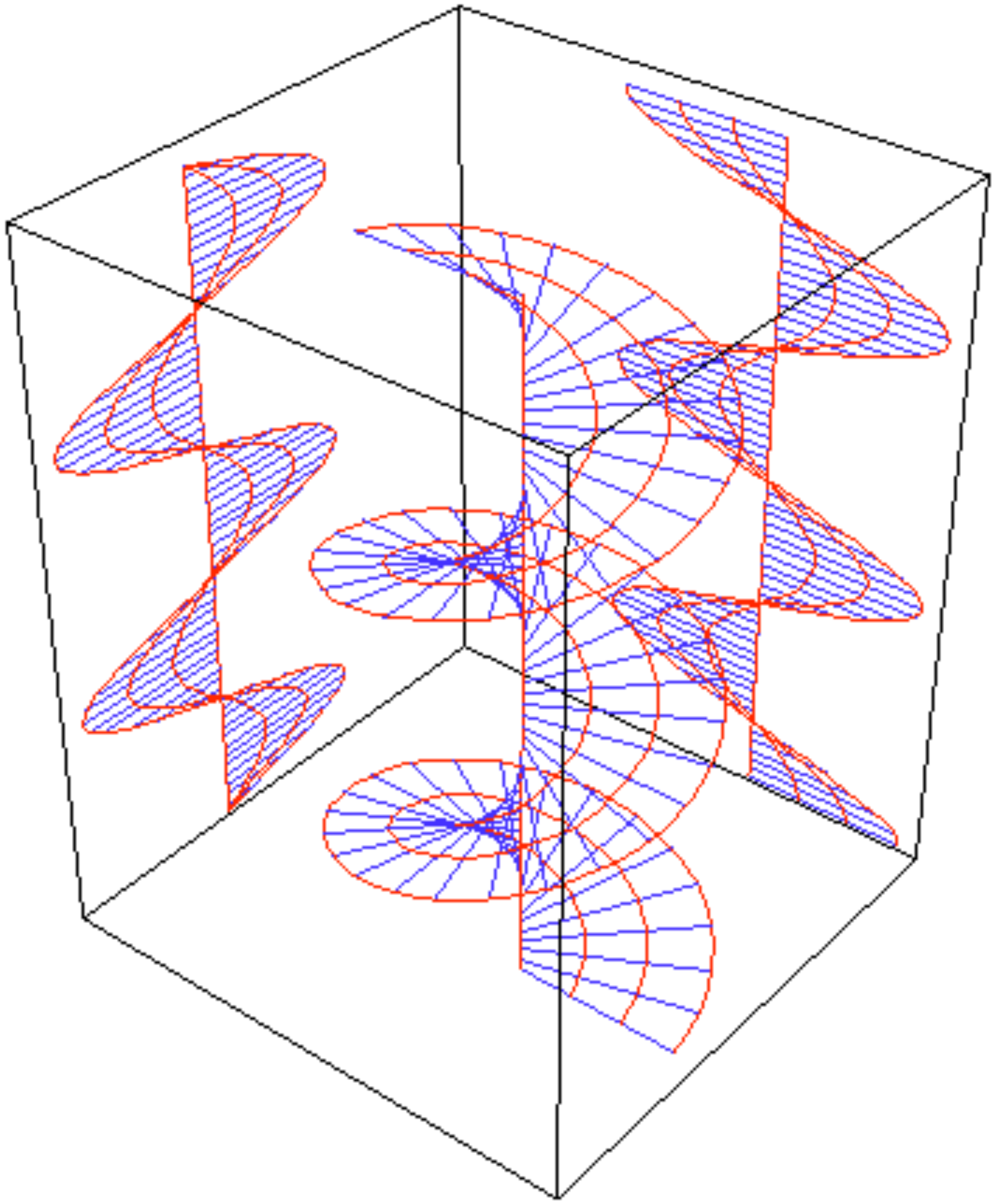
Angle Cosine thm: $\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c.$

For triangles with small edge lengths a, b, c the spherical formulas become the planar ones if all terms that are at least cubic in the edge lengths are ignored, i.e.

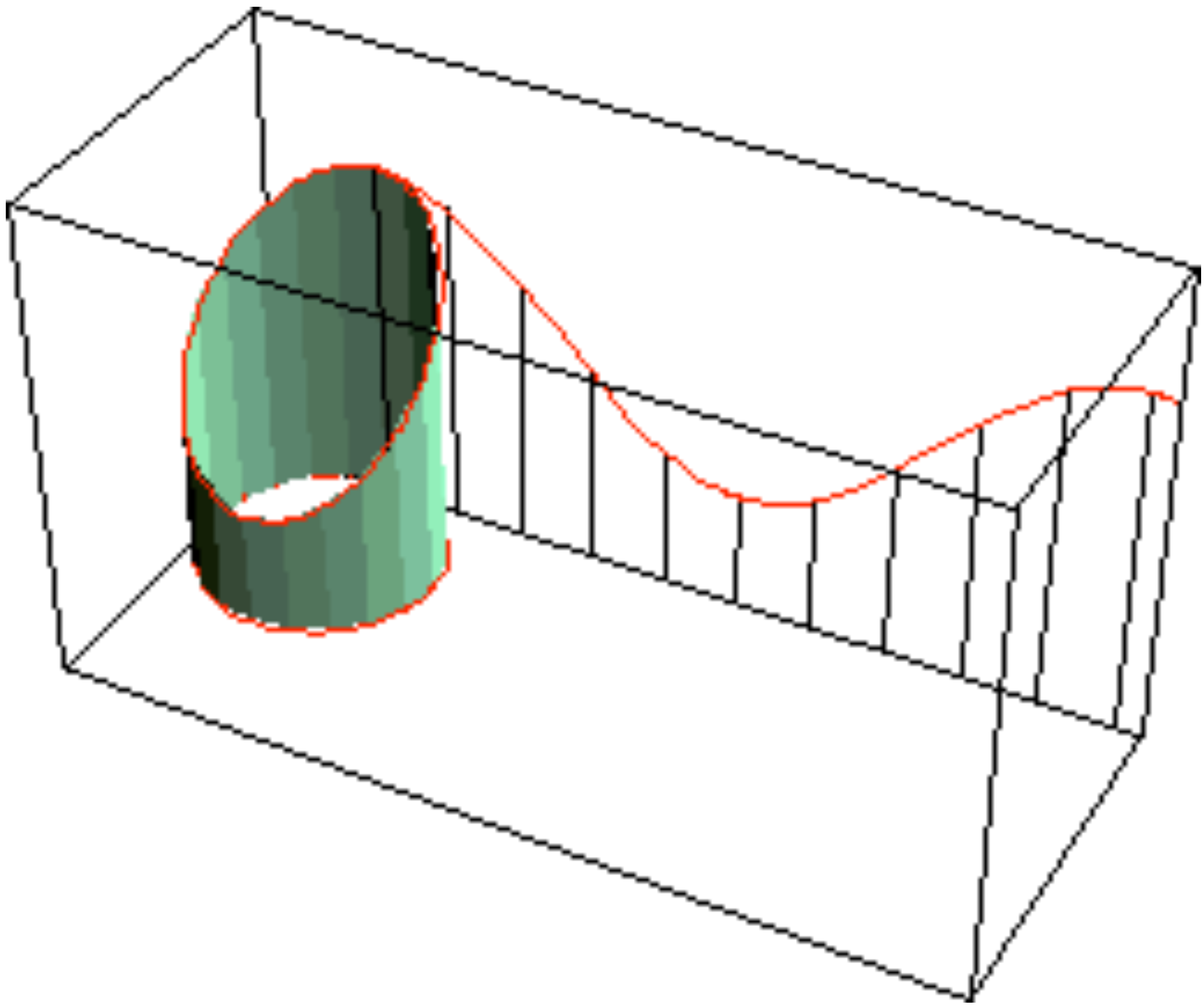
$$\sin a \approx a,\ \cos a \approx 1 - a^2/2,\ \sin a \cos c \approx a.$$

Trivia

Orthogonal Projections of the Helix



The sinusoid is an orthogonal projection of the helix space curve. In 3DXM, a helix can be seen in the Space Curves category and, independently, via the Action Menu entry `Show Planar Curve As Graph`, after selecting `Circle`.



The sinusoid is the development of an obliquely cut right circular cylinder—i.e., the edge of the cylinder rolls out to a sinusoid.

Numerical Computations

From the ODEs one immediately gets the Taylor series ($\lim_{N\to\infty}$ below) which are convenient to compute sine and cosine for small arguments (so that small N are sufficient):

$$\sin x \approx \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad \cos x \approx \sum_{k=0}^{N} \frac{(-1)^k}{(2k)!} x^{2k}.$$

Numerical efforts can be greatly reduced with the simple angle tripling formulas

$$P_3(y) := 3y - 4y^3,$$
$$\sin(3x) = 3\sin(x) - 4\sin^3(x) = P_3(\sin x),$$
$$\cos(3x) = -3\cos(x) + 4\cos^3(x) = -P_3(\cos x).$$

Accuracy, already with $N = 2$, is surprising:
$0 = \sin\pi \approx P_3(P_3(P_3(Taylor_5(\pi/27)))) \approx -1.6 \cdot 10^{-9}$.

Since $\sin x \approx x$ is a very good approximation for small x we get $\lim_{n\to\infty} P_3^{\circ n}(x/3^n) = \sin(x)$.

Or, *much faster:* $\lim_{n\to\infty} P_3^{\circ n}(Taylor_5(x/3^n)) = \sin(x)$.

Since $\sin'(\pi) = -1$, one can compute π with the Newton iteration $x_{n+1} := x_n + \sin(x_n),\ x_0 = 3$.

This can also be seen as a consequence of $\arcsin(x) \approx x$ for small x and $\pi/2 < x < \pi \Rightarrow \pi = x + \arcsin(\sin(x))$.
Then, using the better approximation:

$$\arcsin(x) \approx x + \tfrac{1}{6}x^3,$$

one gets the really fast iteration

$$x_{n+1} := F(x_n) := x_n + \sin(x_n) + \tfrac{1}{6}\sin(x_n)^3.$$

Example: $F(2.5) = 3.1342,\ F(F(2.5)) = 3.14159265358$.

When Archimedes estimated π, he used the inverse of the angle doubling formulas to compute the length of a regular inscribed 2n-gon from the length of a regular n-gon, namely

$$\cos(\frac{x}{2}) = \sqrt{0.5(1+\cos(x))},\ \sin(\frac{x}{2}) = 0.5\sin(x)/\cos(\frac{x}{2}).$$

XL, HK.