Untangle double twist in SO(3) *

The goal of this animation is first to visualize closed curves in SO(3) and then morph these closed curves to visualize homotopies. While a full rotation around a fixed axis is a nontrivial loop in SO(3), doing it twice is a nullhomotopic loop. It was made famous outside mathematics by Dirac's belt trick and Feynman's plate trick.

Bob Palais has two descriptions of the homotopy in SO(3) which are much simpler than to look at a 2-parameter family of orthogonal matrices:

(a) A rotation with axis vector $\vec{a}$ and rotation angle φ can be obtained as composition of two 180° rotations around axes $\vec{b}, \vec{c} \perp \vec{a}$ with angle$(\vec{b}, \vec{c}) = \varphi/2$. To get the double twist around the z-axis (the loop of the above tricks) choose

$$\vec{b} := (1, 0, 0),\ \vec{c}(s) := (\sin(s), \cos(s), 0),\ 0 \le s \le 2\pi.$$

It is obvious how to homotop the equator $s \mapsto \vec{c}(s)$ into the point loop $\vec{b}$, hence homotop the double twist into id.

(b) A homotopy from one full twist to its inverse is given by rotating the axis vector to its negative - simple enough. Composition of this homotopy with the chosen full twist is a homotopy from a double twist to the id-loop.

In other words, the challenge is not to formulate such a homotopy, but to visualize it. And, again, the challenge is to visualize a loop in SO(3), because a homotopy is simply a time dependent morph of a loop.

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

Let $A(s) \in SO(3)$ be a loop of rotations with $A(0) = \mathrm{id} = A(1)$. Consider a family of concentric spheres. They should really be all the same spheres, but we use the radius parameter similar to the graph representation of a 1-dimensional function. Draw a spherical polygon P on the innermost sphere of radius 1. Then draw the spherical polygons $A(s)P$ on the spheres of radius $1 + s$. By following the position of the moving spherical polygon as s increases, one gets a good impression of the family $A(s)$ of rotations.
We use two opposite spherical polygons instead of one, to emphasize the fixed innermost sphere.
The default morph $0 \le aa \le 1$ unwraps the double twist, of course keeping the endpoints fixed.
The morph $0 \le bb \le 1$ with $aa = 0$ creates the double twist around the z-axis (a loop only if $bb = 1$).
The circum radius of the spherical polygons is cc.
One can choose with $dd \in \{3, 4, 5, 6\}$ the number of vertices of the spherical polygon ($dd = 5.2$ is a pentagon star).
In the Action Menu is an entry intended for this demo:

`Reflect tube in ImagePlaneYdirection`

This allows to change the position of the spherical polygon. The default position is around the z-axis. Moving it away from the axis towards the equator creates rather different images of the same family $A(s)$ of rotations. Clicking this entry a second time returns to the original position - unless one has rotated the object with the mouse.
H.K.