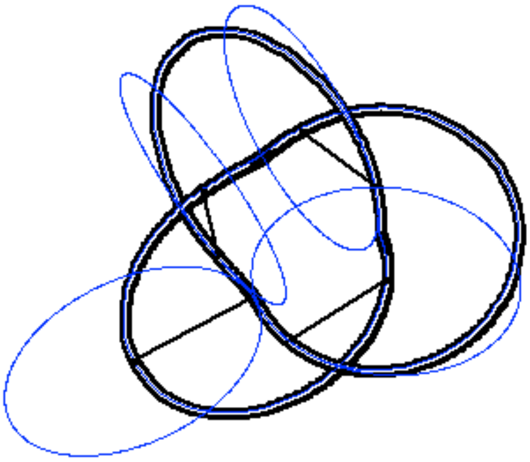
Autoevolutes*

Or: Closed constant curvature space curves which are their own evolutes



2 - 3 - Autoevolute with some osculating circles

See also: Space curves of constant curvature

The suggestion to look for these spectacular curves is from Ekkehard Tjaden, the details are joint work.
Assume that the curve c is parametrized by arclength s, has constant curvature κ and its Frenet frame is $\{T(s),\ N(s),\ B(s)\}$. Its evolute is

$$\tilde{c}(s) = c(s) + \frac{1}{\kappa} \cdot N(s),\ \tilde{c}\,'(s) = \frac{\tau(s)}{\kappa} \cdot B(s).$$

The velocity of the evolute is therefore $|\tilde{c}\,'(s)| = \frac{|\tau(s)|}{\kappa}$ and its Frenet frame is

$\tilde{T}(s) = \mathrm{sign}(\tau)B(s),\ \tilde{N}(s) = -N(s),\ \tilde{B}(s) = \mathrm{sign}(\tau)T(s)$.
That the evolute has the same constant curvature κ follows from: $\tilde{N}'(s) = -N'(s) = \kappa \cdot T(s) - |\tau(s)| \cdot \tilde{T}(s)$. For the torsion $\tilde{\tau}(s)$ of the evolute differentiate $\tilde{B}(s)$ with respect to arclength of the evolute: $\mathrm{sign}(\tau)\tilde{B}'(s) = T'(s) = -\kappa\tilde{N}(s)$, hence $\tilde{\tau}(s) = \mathrm{sign}(\tau)\kappa/|\tilde{c}\,'(s)| = \kappa^2/\tau(s)$.

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

The two curves $c, \tilde{c}$ are geodesics on the canal surface which envelops the spheres of radius $1/2\kappa$ with midpoints on the curve $m(s) = (c(s) + \tilde{c}(s))/2$, because their principle normals $N, \tilde{N}$ are orthogonal to the canal surface. This canal surface is, in the case of the example above, a wobbly torus - maybe that helps the visualization.

Constant curvature curves in general do not have congruent evolutes and even if $c, \tilde{c}$ are congruent, this is difficult to check in case c is parametrized by arc length. One may look for curves with non-constant velocity $v(t) = |\dot{c}(t)|$, choose $\tau(t)$ in terms of $v(t)$ and try to arrange things so that the first half of the curve is congruent to the evolute of the second half and vice versa. This is achieved by the following version of the Frenet equations which Ekkehard suggested. It assumes $t \in [0, 2\pi]$ and $v(t+\pi) = 1/v(t)$.

$$\begin{aligned}
\dot{T}(t) &= \qquad\qquad +\kappa \cdot v(t) \cdot N(t), \\
\dot{N}(t) &= -\kappa \cdot v(t) \cdot T(t) + \kappa/v(t) \cdot B(t), \\
\dot{B}(t) &= \qquad\qquad -\kappa/v(t) \cdot N(t).
\end{aligned}$$

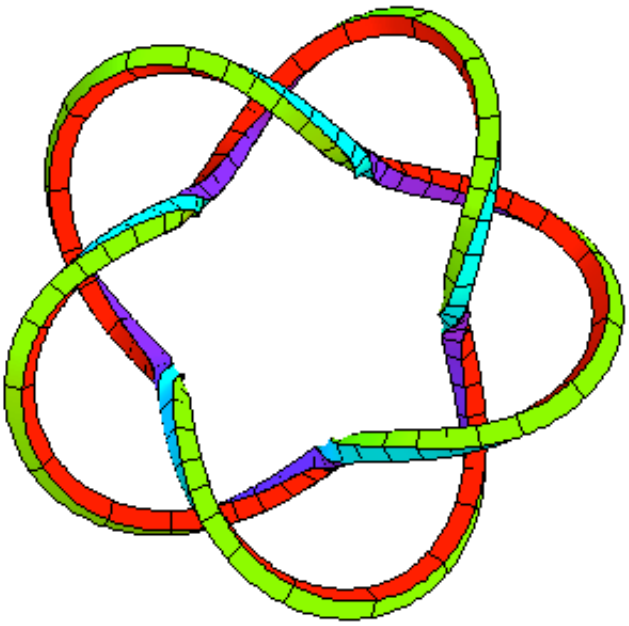
In 3DXM we choose first $h(t) = bb \cdot (cc \sin(t) + ee \sin(3t))$ and then $v(t) = \sqrt{1 + h(t)^2} - h(t)$ or also $v(t) = \exp(h(t))$; as with the other constant curvature curves $\kappa = aa$. Comparison with the standard Frenet equations shows

$$\tau(t) = \kappa/v(t)^2.$$

Note that $v(t)$ and $\tau(t)$ are, relative to $t^* = \pi/2 + n\pi$, even functions. This implies that the principle normals at these points are symmetry normals, saying that 180° rotation around these normals map the curve onto itself. Closed

examples can therefore be found by solving a 2-parameter problem:
a) get the symmetry normals to lie in a plane – they then automatically intersect in one point.
b) get neighbouring symmetry lines to intersect with a rational angle - preferably angles like $\pi/2, \pi/3, 2\pi/3 \ldots$.

This looks similar to the case of single constant curvature curves. However in that case the constant Fourier term of the torsion is a parameter that allows to solve problem a) for all choices of other parameters. This allows to deal with problem b) assuming that a) is already solved.
For the autoevolutes we have not found such a simplifying parameter. By trial and error one has to close the curve to pixel accuracy. Only then does the automatic solution of the 2-paramter problem work and produce a high accuracy solution.



Note that the curve has large torsion – i.e. twists rapidly and moves slowly – opposite to the points with small torsion, where the curve is fairly circular and its velocity is large.

H.K.