The Pseudosphere*
from a Sine-Gordon solution

The Pseudosphere was first found as a surface of revolution, with the Tractrix as meridian (see Planar Curves). It has Gauss curvature $K = -1$. See:
`Constant Curvature Surfaces of Revolution`.

Later in the 19th century it was discovered that surfaces with $K = -1$ can be constructed from soliton solutions of the Sine-Gordon Equation (SGE). This is explained in:
`About Pseudospherical Surfaces,`
which can be obtained from the Documentation Menu.

At about the same time, in 1868, Beltrami proved that the axiomatically constructed non-Euclidean geometry of Bolyai and Lobachevsky was the same as the simply connected 2-dimensional Riemannian geometry of Gauss curvature $K = -1$; for example the Riemannian metric of the Pseudosphere, extended to the plane: $du^2 + \exp(-2u)dv^2$. Their common name today is *Hyperbolic Geometry*.

The meridians are examples of *asymptotic geodesics*, a key notion in hyperbolic geometry. Curves, orthogonal to a family of asymptotic geodesics are called *horocycles* in hyperbolic geometry. They have infinite length in the simply connected case, on the Pseudosphere one sees finite por-

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http://3D-XplorMath.org/

tions as the latitude circles.

In the theory which relates SGE solutions to surface in $\mathbb{R}^3$ of Gauss curvature $K = -1$, one first writes down the first and second fundamental forms in terms of such a solution $q(x,t)$ of SGE:

$$\mathrm{I} = dx^2 + dt^2 + 2\cos q\ dx\,dt, \quad \mathrm{II} = 2\sin q\ dx\,dt,$$

The Gauss-Codazzi integrability conditions are satisfied, because q is an SGE solution. The Gauss curvature is the quotient of the determinants of the two forms, i.e. $K = -\sin^2(q)/\sin^2(q) = -1$. One then obtains the first parameter line of the surface by integrating an ODE and the transversal other family by integrating a second ODE. The first and second fundamental forms above are written in asymptote coordinates, which means: the normal curvature of the surface in the direction of the parameter lines is 0. (Note that x and t are arc length parameters on the parameter lines. This leads to the Tchebycheff net mentioned in "About Pseudospherical Curves".) Such parametrizations do not offer a good view of the surface. In 3DXM, therefore, the integration first creates one curvature line of the surface and secondly the orthogonal family of curvature lines with $u = x + t, v = x - t$. One can view the integration before the surface is shown, with these parameter lines.

The SGE solution for the Pseudosphere is:

$q(x,t) := 4\arctan(\exp(x{+}t)),\ \ qc(u,v) = 4\arctan(\exp(u)).$

H.K.