$(K=-1)$ - Surfaces from SGE solitons *

In 3DXM there are several explicit solutions $q(u,v)$ of the Sine-Gordon Equation (SGE: $q_{uu} - q_{vv} = sin(q(u,v))$). They use Parameters as follows:

`Pseudosphere`: No parameters, $q(u,v) = 4\arctan(\exp(u))$

`One-Soliton (or Dini family or Kink solutions)`: $0 < aa < 1$, used for the so called Dini deformation: $q(u,v) = 4\arctan(\exp(u/\sin(\pi aa) + v/\tan(\pi aa)))$

`Two-Soliton`: $0 < aa, bb < 1$. The limit $aa = bb$ is included, because it is a deformation of the Kuen surface.

`Three-Soliton`: $0 < aa, bb, cc < 1$, no equal parameters.

`Four-Soliton`: $0 < aa, \ldots, dd < 1$, no equal parameters.

`Periodic SGE solutions`: $f''(u) = -\frac{cc}{aa} * \sin(aa \cdot f(u))$, $aa = 1, cc = -1, f(0) = hh, f'(0) = bb,\ 0 < dd < \infty$. (Either $bb = 0$ for even, or $hh = 0$ for odd solutions.) $q(u,v) = f(\frac{(dd+1/dd)}{2} \cdot u + \frac{(dd-1/dd)}{2} \cdot v)$, $dd = 1$ for rotational symmetry, try: $hh = 1, dd = 1.35$.

`Breather`: $0 < aa \leq 1,\ bb = cc = 0$, ($aa = 1$ allowed) $q(u,v) = 4\arctan(\frac{aa}{\cosh(aa\cdot u)} \cdot \frac{\sin(\sqrt{(1-aa^2)}\cdot v)}{\sqrt{(1-aa^2)}}$

If $cc = 0$ and $bb > 0$ integer, then 3DXM chooses aa so that the breather closes up with order bb rotation symmetry. Finally $cc > 0$ applies the Dini deformation to this breather.

`Double Breather`: $aa, cc \neq 0$. Coded to do the complex 2-Soliton. Currently: $z = aa + i \cdot cc, \bar{z} = aa - i \cdot cc$.

---

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

Each $q(u,v)$ determines a pair of first and second fundamental forms, in asymptote parameters $x = \frac{u+v}{2}, t = \frac{u-v}{2}$:

$$\mathrm{I} = dx^2 + dt^2 + 2\cos q \; dx\, dt, \quad \mathrm{II} = 2\sin q \; dx\, dt,$$

for which the Gauss-Codazzi integrability conditions are satisfied (because they simplify to q being an SGE solution $q_{xt} = \sin(q(x,t))$. This says:
There are parametrized surfaces in $\mathbb{R}^3$ with these first and second fundamental forms. They have Gauss curvature $K = -1 = \det(\mathrm{II})/\det(\mathrm{I})$. The parameter lines are asymptote lines on these surfaces and x, t are arclengths on the parameter lines. For more details see:

`About Pseudospherical Surfaces` ,
available from the Documentation Menu.

Since pictures of surfaces, drawn with asymptote line parametrization, do not give a good 3D-impression of the surface, 3DXM uses instead curvature line parametrizations, i.e. $u = x + t, v = x - t$ with the corresponding SGE solution $qc(u,v) := q((u+v)/2, (u-v)/2)$. 3DXM first computes one curvature line as solution of an ODE. Next the family of orthogonal curvature lines is drawn. This determines the parametrized surface. It is then rendered according to the viewer's choice.

Recall the definition cosec $:= 1/\sin$.

The **One-Soliton** solution of SGE ($s + \frac{1}{s} = 2\mathrm{cosec}(aa\pi)$):
$q(x,t) = 4\arctan(\exp(\ s\cdot x + 1/s \cdot t\ ))$, in curvature lines:
$qc(u,v) = 4\arctan(\exp(\ \mathrm{cosec}(aa\cdot\pi)\cdot u + \mathrm{cotan}(aa\cdot\pi)\cdot v\ ))$.
The first drawn curvature line is a planar curve.

For $s = 1,\ aa = 1/2$ the solution qc depends only on u, the corresponding surface is the Pseudosphere. Vice versa, for every surface of revolution the function qc depends only on u and the SGE reduces to the ODE $f''(u) = +\sin(f(u))$. The even solutions of this ODE correspond to the hyperbolic ($K = -1$) surfaces and the odd solutions give the surfaces of revolution with finite cone singularities. The only elementary solution is $f(u) = 4\arctan(\exp(u))$ with $f(0) = \pi, f'(0) = 2$ and the cone singularity at infinity.

While this first step is obtained as explicit integration of the ODE $f''(u) = \sin(f(u))$, the next steps follow from a result of Bianchi, reformulated for SGE solutions:
If a pair of functions $q(x,t), Q(x,t)$ satisfy the pair of ODEs:

$$Q_x(x,t) = +q_x(x,t) + 2s \cdot \sin(\tfrac{Q+q}{2}(x,t))$$
$$Q_t(x,t) = -q_t(x,t) + 2/s \cdot \sin(\tfrac{Q-q}{2}(x,t))$$

then, for given $q(x,t)$, the system is solvable for $Q(x,t)$ iff $q(x,t)$ is an SGE solution. And then $Q(x,t)$ is also SGE solution and the two functions are called Bäcklund transformations of each other. Moreover, if, for $s_1 \neq s_2$ two such solutions $Q1(x,t), Q2(x,t)$ are given, then a third solution $Qq(x,t)$ is marvelously obtained from

$$\tan(\tfrac{Qq-q}{4}(x,t)) = \tfrac{s_1+s_2}{s_1-s_2} \cdot \tan(\tfrac{Q1-Q2}{4}(x,t)).$$

The 1-Soliton solution gives two such functions $Q1, Q2$ which are Bäcklund transformations of the trivial SGE solution $q(x,t) = 0$ and $Qq(x,t)$ is the following 2-Soliton solution (recall $\tan(\alpha-\beta) = (\tan(\alpha)-\tan(\beta))/(1+\tan(\alpha)\cdot\tan(\beta)))$:

$$Qq(x,t) = 4\arctan\left(\frac{s_1+s_2}{s_1-s_2} \cdot \frac{\exp(s_1\cdot x+1/s_1\cdot t)-\exp(s_2\cdot x+1/s_2\cdot t)}{1+\exp(s_1\cdot x+1/s_1\cdot t)\cdot\exp(s_2\cdot x+1/s_2\cdot t)}\right)$$

This Bianchi generalization also works for complex parameters and one can see directly that for $s_1 = z, s_2 = \overline{z} \neq 0$ the new solution $Qq(x,t)$ is again real. If $|z| = 1$, it is the breather solution below. In 3DXM we use $z = aa + i \cdot cc$. The other (periodic) solutions of $q_{uu} = \sin(q(u))$ are not Bäcklund transformations of known solutions.

The curvature line version, which is needed for visualization, looks more complicated:

The **Two-Soliton** solution of SGE (parameters aa, bb): Define a constant B and functions $A1, A2$ first.

$B := (\cos(bb\,\pi) - \cos(aa\,\pi))/(\cos((aa - bb)\pi) - 1)$,

$A1(u,v) := \mathrm{cosec}(aa\,\pi)u + \mathrm{cotan}(aa\,\pi)v$,

$A2(u,v) := \mathrm{cosec}(bb\,\pi)u + \mathrm{cotan}(bb\,\pi)v$, then put:

$$qc(u,v) := 4\arctan\left(B \cdot \frac{\exp(A1(u,v))-\exp(A2(u,v))}{1+\exp(A1(u,v)+A2(u,v))}\right).$$

Now observe that the 2-Soliton solutions are Bäcklund transformations of the two 1-Soliton solutions which were used in the Bianchi construction!! Therefore the Bianchi construction can be repeated to give the following:

Three-Soliton solution of SGE (params aa, bb, cc): Define three auxiliary functions E, F, H first.

$E(\xi,u,v) := \exp(\ \mathrm{cosec}(\xi\,\pi)u + \mathrm{cotan}(\xi\,\pi)v\ )$,

$$F(\xi_1,\xi_2,u,v) := \frac{\cos(\xi_2\,\pi)-\cos(\xi_1\,\pi)}{\cos((\xi_2-\xi_1)\pi)-1} \cdot \frac{E(\xi_1,u,v)-E(\xi_2,u,v)}{1+E(\xi_1,u,v)\cdot E(\xi_2,u,v)},$$

$$H(\xi_1,\xi_2,\xi_3,u,v) := \frac{\cos(\xi_3\,\pi)-\cos(\xi_2\,\pi)}{\cos((\xi_3-\xi_2)\pi)-1} \cdot \frac{F(\xi_1,\xi_2,u,v)-F(\xi_1,\xi_3,u,v)}{1+F(\xi_1,\xi_2,u,v)\cdot F(\xi_1,\xi_3,u,v)},$$

$$qc(aa, bb, cc, u, v) := 4\arctan(E(aa, u, v)) \\ + 4\arctan(H(aa, bb, cc, u, v)).$$

The **Four-Soliton** solution of SGE (params aa, bb, cc, dd) we give in asymptote coordinates:
Define functions E, F, H (as before) and K first.
$E(s, x, t) := \exp(\ s \cdot x + 1/s \cdot t)$,

$F(s_1, s_2, x, t) := \frac{s_1+s_2}{s_1-s_2} \cdot \frac{E(s_1,x,t)-E(s_2,x,t)}{1+E(s_1,x,t)\cdot E(s_2,x,t)}$,

Recall that $4\arctan(F(s_1, s_2, x, t))$ are 2-Soliton solutions.

$H((s_1, s_2, s_3, x, t) := \frac{s_2+s_3}{s_s-s_3} \cdot \frac{F(s_1,s_2,x,t)-F(s_1,s_3,x,t)}{1+F(s_1,s_2,x,t)\cdot F(s_1,s_3,x,t)}$,

$K(s_1, s_2, s_3, s_4, x, t) :=$
$$\frac{s_3+s_4}{s_3-s_4} \cdot \frac{H((s_1,s_2,s_3,x,t)-H((s_1,s_2,s_4,x,t)}{1+H((s_1,s_2,s_3,x,t)\cdot H((s_1,s_2,s_4,x,t)},$$

Finally

$q(s_1, s_2, s_3, s_4, x, t) :=$
$$4\arctan(F(s_1, s_2, x, t)){+}4\arctan(K(s_1, s_2, s_3, s_4, x, t)).$$
Recall as before $qc(u, v) = q((u + v)/2, (u - v)/2)$.

The **Breather** (soliton) solution of SGE (parameter aa):
Abbreviate $bb := \sqrt{1 - aa^2}$, then

$qc(bb, aa, u, v) := 4\arctan(\ \frac{\sin(bb\,v)}{bb} \cdot \frac{aa}{\cosh(aa\,u)}\ )$.

Recall from above that each of these solutions determines a first and a second fundamental form which satisfy the Gauss-Codazzi integrability conditions. Each parameter line gives a space curve via an ODE which is determined by these fundamental forms. Because of the integrability conditions these space curves fit together to form a surface of Gauss curvature $K = -1$.

H.K.