Note: The following material is adapted from section 3 from the paper "The Genus One Helicoid and the Minimal Surfaces that Led to its Discovery", by David Hoffman, Hermann Karcher, and Fusheng Wei. This paper was originally published in *Global Analysis and Modern Mathematics*, Copyright 1993, Publish or Perish Press.

3. A supply of basic elliptic functions

We give here a self-contained introduction to elliptic functions. The approach is somewhat unconventional. In particular:

(i) The symmetries (or functional equations) of degree-two elliptic functions are usually expressed in terms of Möbius transformations of the Riemann sphere. We want these Möbius transformations to be *isometric* rotations and we will achieve rotation only around coordinate axes of the sphere in $\mathbb{R}^3$, given in $\mathbb{C}$ as $z \to \pm z^{\pm 1}$. In the special case of rectangular or rhombic tori— the ones with "complex conjugation"— we achieve the corresponding Möbius reflections to be *isometric* reflections in coordinate planes (i.e. $z \to \pm \bar{z}$, $z \to 1/\bar{z}$).

(ii) We want to build more complicated functions by multiplying simple ones with known zeros and poles (as we do with rational functions). The classical approach treats only two functions in a distinguished way, namely the ones that go into the equation of the surface. We need a larger such distinguished collection together with their mutual relations.

3.1 Construction of degree-two elliptic functions.

3.1.1. Quotient functions. The typical degree-two elliptic function can be constructed as follows. View the torus T as $\mathbb{C}/\Gamma$. Any 180^0-rotation r of $\mathbb{C}$ around a point c_0 induces an orientation-preserving involution of T with four fixed points (which are given in $\mathbb{C}$ as $c_0 + \frac{1}{2} \cdot \Gamma$, a "halfperiod lattice"). The quotient surface is the Riemann sphere since $\chi(T/r) = 2$. This follows either from $0 = \chi(T) = 2 \cdot \chi(T/r) - 4$ or by applying Euler's $\chi = V - E + F$ to the tessalation of T by the four parallelograms whose vertices are the fixed points of r and to the quotient tessalation, which has $F = 2$ quadrilaterals, $E = 4$ edges and $V = 4$, the fixed points. At this point the quotient sphere is only a conformal sphere, not yet the standard sphere in $\mathbb{R}^3$ which is identified with $\mathbb{C} \cup \{\infty\}$ via stereographic projection; but after we call three arbitrary points of the quotient sphere $0,\ 1,\ \infty$ we have a unique identification with the standard sphere. We then call the quotient map a function. Any two such choices of $0,\ 1,\ \infty$ give two functions which differ by a Möbius transformation. Moreover, if two degree-two elliptic functions f_1, f_2 on T have one branch point $c_0 \in T$ in common then all their branch points agree. Here is a proof. We may assume $f_1(c_0) = 0$, $f_2(c_0) = 0$, and also that f_1 has been constructed as above with a double pole at one of its other three branch points, say c_1. If necessary, replace f_2 by $f_2/(f_2 - f_2(c_1))$ so that $f_2(c_1) = \infty$. Then f_1/f_2 has at most one pole (at c_1) and one zero, hence it is constant, q.e.d.. We may summarize this as follows: Any two degree-two elliptic functions on the same torus differ by a translation of the torus—which positions their branch points— and by a Möbius transformation of the sphere—which positions $0,\ 1,\ \infty$.

3.1.2. Remark. This approach shows that the cross-ratio of the four branch values (in suitable order) depends only on the torus; it is called the *modular invariant* of the torus and it is usually computed from the finite branch values of the Weierstrass $\wp$-function as $(e_1 - e_3)/(e_2 - e_3)$. The differential equations below for our elliptic functions will depend only on this modular invariant.

3.1.3. Symmetries. The following simple observation is responsible for the symmetries of the degree-two elliptic functions: Given one 180^0-rotation r of the torus T , there are three other 180^0-rotations of T which *permute the fixed points* of r . This means for the quotient map $f : T \to T/r$:

$$f \circ \text{ Torusrotation}_k = \text{ Möbiusinvolution}_k \circ f, \qquad k = 1, 2, 3,$$

since on the left side we have degree-two maps $T^2 \to S^2$ with the same branch points as f. One of our aims is to have these Möbius involutions as simple as possible; we will achieve $z \to \pm z^{\pm 1}$.

3.1.4. Special choices. We now use the observations above to construct three functions that have the same distribution of simple zeros and poles as Jacobi's degree-two elliptic functions. A fourth function will be constructed with a double zero and a double pole. On the square torus it is, up to scaling, the Weierstrass $\wp$-function, on other tori it is slightly more different, also the values are changed: $a \cdot \wp + b$. We facilitate our description by choosing a fundamental parallelogram for the torus which has its midpoint at $0 \in \mathbb{C}$

(also called $0 \in T = \mathbb{C}/\Gamma$). The rotational position and the scaling will be specified later. The half-period lattice $0 + \frac{1}{2} \cdot \Gamma$ determines the vertices and the midpoints of the edges of the chosen fundamental parallelogram. To specify a function of degree-two we have to give its branch points in the fundamental parallelogram (a half-period set marked $\diamond$ in the figure) and we have to specify three points that are to go to $0, \infty$ and either 1 or i. It is convenient to write these *values* in the *domain* parallelogram; the choice of their position, of course, determines the functional relations. To simplify building more functions by multiplication we always choose the zeros and poles at the half-period points. By the definition of a quotient map we do not change the values of the (quotient) function if we rotate the torus around the chosen branch points. This means that, for the Jacobi type functions, the branch points have to be chosen as midpoints between the zeros (and hence also as midpoints between the poles). The finite value 1 resp. i is placed at a midpoint between a zero and a pole because 180^0-rotation of the Riemann sphere around 1 interchanges $0, \infty$. This choice is responsible for the simplicity of the Möbius transformation in the functional relations. In the case of the Jacobi type functions and for the geometrically normalized Weierstrass $\wp$-function we denote the 180^0-rotations of the torus and their quotient functions as follows

$$\text{rotations: } r_D, r_E, r_F; r_P \qquad \text{functions: } jd, je, jf; g\wp.$$

The following diagrams *define our four functions*, which, of course, must satisfy $f \circ r = f$.

Diagram From Original Version Omitted Here.

3.1.5. Functional equations. Recall that each of the four rotations permutes the fixed points of the other rotations and that this is the source of the functional equations. In each case we get the first relation from the rotation around the point where the finite value $(1, i)$ was chosen:

$$
\begin{aligned}
\mathit{jd} \circ r_{F} &= \frac{1}{\mathit{jd}}, \quad \text{values } \pm 1 \text{ at fixed points of } r_{F} \\
\mathit{je} \circ r_{F} &= \frac{1}{\mathit{je}}, \quad \text{values } \pm 1 \text{ at fixed points of } r_{F} \\
\mathit{jf} \circ r_{E} &= \frac{-1}{\mathit{jf}}, \quad \text{values } \pm i \text{ at fixed points of } r_{E} \\
g\wp \circ r_{D} &= \frac{-1}{g\wp}, \quad \text{values } \pm i \text{ at fixed points of } r_{D}.
\end{aligned}
$$

This says that the Jacobi type functions are odd, $g\wp$ is even:

$$j \circ r_{P} = -j, \quad g\wp \circ r_{P} = g\wp .$$

The diagrams, completed with these first special values, look like this:

Diagram From Original Version Omitted Here.

We read off the next relations which (3.1.3) promised:

$$\mathit{jd} \circ r_{E} = \frac{-1}{\mathit{jd}}, \quad \mathit{je} \circ r_{D} = \frac{-1}{\mathit{je}}, \quad \mathit{jf} \circ r_{D} = \frac{+1}{\mathit{jf}}$$

and we marked the fixed points of the used rotations in the diagram by $\times$. At these fixed points ($\times$) the functions have *values* which are fixed under the corresponding Möbius involution $j \to \pm 1/j$, namely ± 1, resp. $\pm i$. The distribution of the $\pm$signs is a matter of orientation. For example in the case of jd the parallelogram disk around $0 \in T$ that is indicated in the figure is mapped by jd biholomorphically to a disk around $0 \in \mathbb{C}$ that has $1,\ i,\ -1,\ -i$ on its boundary in the positive order. Therefore we have these values in the same order around $0 \in T$. The same argument applies to $\mathit{je},\ \mathit{jf}$.

3.1.6. Branch values. So far we have not seen anything that is specific to the torus under consideration. We have already mentioned that the cross-ratio of the branch values is the modular invariant which distinguishes the tori. So, in each of the three Jacobi type functions, $jd\,, je, jf$, we first give one branch value a name: D, E, resp. F; then since all the functional equations (3.1.3) were shown to use the Möbius involutions $z \to \pm z^{\pm 1}$ we find the other branch values as $\pm(D, E, F)^{\pm 1}$. We summarize by giving the fundamental parallelograms with all the special values. The subparallelogram with the first named branch point in its right upper corner is shaded. Rotation around a zero or pole sends a branch value B to $-B$, rotation around points with value ± 1 sends B to $1/B$ and rotation around $\pm i$ sends B to $-1/B$.

Diagram From Original Version Omitted Here.

3.1.7. Relations modulo translations. To emphasize the close relation between these functions we also give the Möbius transformations that transform these functions, modulo torus translations, one into another. We choose the translations that map one of the shaded subparallelograms to another one. The translated functions have the same branch points and are therefore Möbius-related:

$$je \circ \text{translation}_1 = \frac{jd\, -1}{jd\, +1}, \quad jf \circ \text{translation}_2 = i \cdot \frac{jd\, -i}{jd\, +i},$$
$$jf \circ \text{translation}_3 = -\frac{je -i}{je +i}.$$

In particular, this shows the relation of the branch values:

$$E = \frac{D-1}{D+1}, \quad F = i \cdot \frac{D-i}{D+i}, \quad F = -\frac{E-i}{E+i}.$$

Finally we observe that $g\wp$ and $je \cdot jf$ have the same zeros and poles and agree where $g\wp = i$. This gives the geometrically normalized Weierstrass $\wp$-function.

Its Möbius relation with jd and its diagram of special values are as follows:

$$g\wp \circ \text{translation}_4 = \text{möbius}_4(jd) := -i \cdot \frac{jd + D}{jd - D}, \quad g\wp = je \cdot jf$$

with one finite branch value given as $P = \text{möbius}_4(1/D) = i(D^2+1)/(D^2-1)$.

Diagram From Original Version Omitted Here.

3.1.8. Remark. We compute for later use the cross-ratio of the branch values:

$$1 - 4(D^2 + D^{-2} - 2)^{-1} = (E^2 + E^{-2} + 2)/4 = -4(F^2 + F^{-2} - 2)^{-1} = -P^2$$

and recall once more that it is the classical modular invariant of the torus.

3.2 Functional relations between the four functions.

3.2.1. Biquadratic Relations. The most common description of a Riemann surface is in terms of an algebraic relation between two functions (usually called z, w) on the Riemann surface. One may interpret the functions as (local) coordinates — away from their branch points of course—and the algebraic relation is a description of the change of coordinates. We can take any pair of our degree-two elliptic functions and find a biquadratic relation between them. The following list of relations is immediately verified since both sides have the same zeros, the same poles and agree at another point (a branch point of the function involved on the left).

$$\begin{aligned}
\mathit{jd} - \frac{1}{\mathit{jd}} &= \frac{D - 1/D}{2i}(\mathit{je} - \frac{1}{\mathit{je}}) = \frac{2i}{E - 1/E}(\mathit{je} - \frac{1}{\mathit{je}}) \\
\mathit{jd} + \frac{1}{\mathit{jd}} &= \frac{D + 1/D}{2}(\mathit{jf} + \frac{1}{\mathit{jf}}) = \frac{2}{F + 1/F}(\mathit{jf} + \frac{1}{\mathit{jf}}) \\
\mathit{je} + \frac{1}{\mathit{je}} &= \frac{2}{F - 1/F}(\mathit{jf} - \frac{1}{\mathit{jf}}) = \frac{E + 1/E}{2i}(\mathit{jf} - \frac{1}{\mathit{jf}}) \\
\frac{D^2}{\mathit{jd}^2} &= \frac{g\wp - 1/g\wp - P + 1/P}{2i - P + 1/P} = D^2 \cdot \frac{g\wp - 1/g\wp - P + 1/P}{F - 1/F - P + 1/P}
\end{aligned} \tag{3.5}$$

As an example, let the branch value D be given and rename $w = \mathit{jd}, z = \mathit{je}$. The first equation then looks more familiar

$$(w - \frac{1}{w})/(D - \frac{1}{D}) = (z - \frac{1}{z})/(2i),$$

but we cannot immediately recover all our information about jd, je from this equation, and of course, there are no relations with other functions.

3.2.2. Logarithmic derivatives. Another common choice of a pair of functions to describe a torus is to take a degree-two elliptic function together with its logarithmic derivative. In the following list of relations the functions on both sides have the same zeros and poles; moreover, the first term of their Laurent expansion at $0 \in \mathbb{C}$ is the same — hence they agree. First we express the logarithmic derivative of the Jacobi type functions in terms of other functions; then we differentiate $g\wp = \mathit{je} \cdot \mathit{jf}$; we also note that $g\wp + 1/g\wp$ is a derivative; finally we give the differential equations:

$$\begin{aligned}
\frac{\mathit{jf}'}{\mathit{jf}} &= \mathit{jd}'(0) \cdot (\frac{1}{\mathit{jd}} - \mathit{jd}) = \mathit{jd}'(0) \frac{-2i}{1/E - E} \cdot (\frac{1}{\mathit{je}} - \mathit{je}) \\
\frac{\mathit{je}'}{\mathit{je}} &= \mathit{jd}'(0) \cdot (\frac{1}{\mathit{jd}} + \mathit{jd}) = \mathit{jd}'(0) \frac{2}{1/F + F} \cdot (\frac{1}{\mathit{jf}} + \mathit{jf}) \\
\frac{\mathit{jd}'}{\mathit{jd}} &= \mathit{je}'(0) \cdot (\frac{1}{\mathit{je}} + \mathit{je}) = \mathit{jf}'(0) \cdot (\frac{1}{\mathit{jf}} - \mathit{jf}) \\
\frac{g\wp'}{g\wp} &= \frac{\mathit{je}'}{\mathit{je}} + \frac{\mathit{jf}'}{\mathit{jf}} = \mathit{jd}'(0) \cdot \frac{2}{\mathit{jd}} \\
\left(\frac{1}{\mathit{jd}}\right)' &= -\frac{\mathit{jd}'}{\mathit{jd}^2} = \frac{g\wp''(0)}{2\,\mathit{jd}'(0)} \cdot (g\wp + \frac{1}{g\wp})
\end{aligned}$$

3.2.3. Differential equations. The three Jacobi type functions have the same differential equation in terms of one of their branch values B (Recall that $B^2 + B^{-2}$ can be expressed by the modular invariant.

$$\left(\frac{j'}{j}\right)^2 = j'(0)^2 \cdot (j^2 + j^{-2} - B^2 - B^{-2})$$
$$\left(\frac{g\wp'}{g\wp}\right)^2 = -2\, g\wp''(0) \cdot (g\wp - \frac{1}{g\wp} - P + \frac{1}{P})$$

We repeat that these relations hold because both sides have the same zeros, the same poles and their Laurent expansions at $0 \in \mathbb{C}$ agree. Note, that at this point the derivatives at $0 \in \mathbb{C}$ in the above relations are not yet determined because we have not fixed the scaling size and the rotational position of the fundamental parallelogram in $\mathbb{C}$. If we fix in *one* of the differential equations this derivative at 0 then the size and rotational position of the fundamental domain and also the derivative at 0 of each of the other functions are chosen. Their relation is obtained by comparing in the above biquadratic equations (3.2.1) Laurent expansions at $0 \in \mathbb{C}$:

$$\begin{aligned}
je'(0) &= \frac{D - 1/D}{2i} \cdot jd'(0) = \frac{2i}{E - 1/E} \cdot jd'(0) \\
jf'(0) &= \frac{D + 1/D}{2} \cdot jd'(0) = \frac{2}{F + 1/F} \cdot jd'(0) \\
jf'(0) &= \frac{-2}{F - 1/F} \cdot je'(0) = \frac{E + 1/E}{-2i} \cdot je'(0) \\
g\wp''(0) &= \frac{2}{P + 1/P} \cdot (jd'(0))^2
\end{aligned}$$

(For the last line insert $D^2 = (P + i)/(P - i)$ into $-D^2 \cdot (2i - P + 1/P) = D^2 \cdot (P - i)^2/P$.)

3.3 Specializations: Rectangular and Rhombic Tori.

3.3.1. Reflection symmetries. The tori with orientation reversing symmetries are known as tori with complex conjugation. The ones which are quotients of $\mathbb{C}$ by rectangular lattices (basis $\{1, i \cdot t\}$) are called *rectangular tori*, the ones with a lattice basis of equal length ($\{1, e^{i\varphi}\}$) are called rhombic tori. They are easy to distinguish: For the rectangular tori the axis of reflection in $\mathbb{C}$ is parallel to the edges of a rectangular fundamental domain and it projects to a fixed point set on the torus having *two* components; for the rhombic tori the axis of reflection in $\mathbb{C}$ is parallel to a diagonal of the rhombic fundamental domain and it projects to a fixed point set on the torus having *one* component. Now we assume that the branch points of a degree-two elliptic function have been chosen on the torus, then only four of the mentioned reflections permute the branch points: the axis of the reflection has either to pass through branch points or through midpoints between branch points. Because the branch points—i.e. the fixed points of the 180^0-rotation by which we divide to get the degree-two function — are permuted by the reflections of the torus, these orientation reversing involututions pass to the sphere and we get further symmetries of our functions. We can determine these as Möbius reflections in coordinate planes (i.e. $z \to \pm\bar{z}, z \to 1/\bar{z}$), because the fixed point set of the reflection passes through points at which we chose simple antipodal values of the function, namely ($\in \{0, \infty, \pm 1, \pm i\}$).

3.3.2. Rectangular Tori. The image under jd, je of the symmetry lines joining points with values $0, 1, \infty$ is the real line. It is therefore reasonable

to normalize $\mathit{jd}'(0) = 1$,

because then $\mathit{jd}, \mathit{je}, \mathit{jf}$ map the real resp. the imaginary axis and the respectively parallel boundaries of the fundamental rectangle to the real resp. imaginary axis. The remaining symmetry lines are mapped to the unit circle, in particular $D \in \mathbb{S}^1$. Mainly we will be interested in rhombic tori, but for illustrative purposes we first specialize our formulas to the rectangular case. The branch values are in the rectangular case (computed from D) as follows:

$$D := e^{i\alpha},\ E = \frac{e^{i\alpha-1}}{e^{i\alpha+1}} = i\tan\alpha/2,\ F = \frac{\cos\alpha}{1+\sin\alpha},\ P = i \cdot \frac{e^{2i\alpha+1}}{e^{2i\alpha-1}} = \cot\alpha.$$

Each of the following differential equations (and also the equation between $\mathit{jd}, g\wp$) describe the torus in terms of its modular invariant $-\cot(\alpha)^2$

$$\left(\frac{\mathit{jd}'}{\mathit{jd}}\right)^2 = (\mathit{jd}^2 + \frac{1}{\mathit{jd}^2} - 2\cos 2\alpha)$$

$$\frac{1}{4}\left(\frac{g\wp'}{g\wp}\right)^2 = -\frac{\sin 2\alpha}{2} \cdot (g\wp - \frac{1}{g\wp} - 2\cot 2\alpha) \quad = \quad \frac{1}{\mathit{jd}^2}$$

The *square torus* has the 45^0-diagonals as additional symmetry lines, hence

$$\alpha = \pi/4, P = 1, g\wp''(0) = \sin 2\alpha = 1.$$

3.3.3. Rhombic Tori. We view these tori as deformations of the square torus which preserve the *diagonal* symmetries. Let μ denote reflection in one of the diagonals of the rhombic fundamental domain. Then we have

$$\overline{jd \circ \mu} = i \cdot jd\,, \quad \overline{g\wp \circ \mu} = -\, g\wp\,.$$

This says that on the diagonals we have $jd \in e^{\pm i\pi/4} \cdot \mathbb{R}$, $g\wp \in i \cdot \mathbb{R}$. The normalization $jd\,'(0) = 1$ therefore implies that the diagonals of the rhombic fundamental domain point in the 45^0-directions—a reasonable rotational normalization. Furthermore we have for the branch values $\overline{D} = -i \cdot D, \overline{P} = 1/P$, and this gives us the branch value parametrization of rhombic tori via differential or functional equations:

$$\begin{aligned}
D &= R \cdot e^{i\pi/4}, \quad P = e^{i\rho}, \qquad \text{related via} \\
D^2 &= iR^2 = \frac{P+i}{P-i} = \frac{i \cdot \cos\rho}{1 - \sin\rho} = i \cdot \cot(\pi/4 - \rho/2) \\
\left(\frac{jd\,'}{jd}\right)^2 &= (jd^{\,2} + \frac{1}{jd^{\,2}} - 2i \cdot \tan\rho) \\
\frac{1}{4}\left(\frac{g\wp'}{g\wp}\right)^2 &= -\frac{1}{2\cos\rho} \cdot (g\wp - \frac{1}{g\wp} - 2i \cdot \sin\rho) \quad = \quad \frac{1}{jd^{\,2}}.
\end{aligned}$$

Our functions have more symmetries since we have two more reflections that permute the branch points. The above μ fixed the zeros of jd (where $g\wp = 0, \infty$) and permuted the poles; let now ν be one of the reflections that fixes the poles of jd (where $g\wp = e^{i\rho}, -e^{-i\rho}$) and permutes the zeros. Then

$$\begin{aligned}
\overline{g\wp \circ \nu} &= \frac{1}{g\wp}, \quad \text{i.e. } g\wp \in \mathbb{S}^1 \text{ on the fixed point set of } \nu \\
\overline{jd \circ \nu} &= i \cdot jd\,, \quad \text{i.e. } jd \in e^{\pm i\pi/4} \cdot \mathbb{R} \text{ on the fixed point set of } \nu.
\end{aligned}$$

For the other two functions je, jf we have relations such as $\overline{je \circ \mu} = -i \cdot jf$, $\bar{E} = i \cdot F$, which we do not use.